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A

TREATISE

ON

ALGEBRA.

BY

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TO

THE COUNCIL

OF

THE UNIVERSITY OF THE CITY OF NEW YORK,

THIS TREATISE

Is respectfully Inscribed,

BY

THEIR OBEDIENT SERVANT,

THE AUTHOR.

PREFACE.

The present Treatise is designed to supply a deficiency which has long been felt. Having been engaged for many years in imparting instruction in Algebra, I have found no text-book entirely adapted to the wants of my pupils. Some treatises, from their extreme conciseness, and their aiming at the utmost rigor of demonstration, are too difficult for the majority of students, while others are far behind the present state of science. The present Treatise was specially designed for the use of the students of New York University, but it is believed to be adapted to the wants of students generally in American colleges. It is designed for youth of fifteen or sixteen years of age, who are supposed to possess ordinary abilities and aptitude for study. I have not, therefore, attempted to demonstrate every principle in the most general and rigorous manner. Such demonstrations would often be unintelligible and repulsive to the majority of students. Nor, on the contrary, have I contented myself with mere mechanical rules. I have aimed to follow the natural order of the youthful mind, deducing general principles from particular examples, subjoining afterward a complete demonstration, whenever it was thought that its force would be clearly apprehended. I have aimed to lead the student to generalize every principle. For this purpose, nearly every problem is twice stated; first in a restricted form, and afterward in a more general one; and I have dwelt upon the interpretation of some peculiar results obtained in the discussion of general problems. It is hoped the student will thus be led to perceive that the solution of particular problems is not the most important province of Algebra.

I have bestowed considerable pains upon the general theory of Equations. Each Proposition is distinctly enunciated, and illustrated by appropriate examples. It is believed that Sturm's Theorem is here exhibited in so simple a form, that youth of ordinary abilities may learn to apply it with facility and pleasure. The admirable method of Horner has been explained at length, and numerous examples are given of its application. The entire work was composed, not exclusively for the best scholars, nor for such as are unwilling to study, but rather for the majority of our students; and as such, it is commended to the attention of teachers of Mathematics.

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ALGEBRA.

SECTION I.

PRELIMINARY DEFINITIONS AND NOTATION.

(*Article* 1.) WHATEVER is capable of increase or diminution is called *magnitude* or *quantity*.

A sum of money therefore is a quantity, since we may increase it or diminish it. A line, a surface, a weight, and other things of this nature are quantities; but an *idea* is *not* a quantity.

(2.) Mathematics is the *science of quantity*, or the science which investigates the means of measuring quantity. The operations of the mind, therefore, such as memory, imagination, judgment, etc. are not subjects of mathematical investigation, since they are not quantities.

(3.) Mathematics is divided into *pure* and *mixed*. Pure mathematics comprehends all inquiries into the relations of magnitude in the abstract, and without reference to material bodies. It embraces numerous subdivisions, such as Arithmetic, Algebra, Geometry, etc.

In the mixed mathematics, these abstract principles are applied to various questions which occur in nature. Thus, in Surveying, the abstract principles of Geometry are applied to the measurement of land; in Navigation, the same principles are applied to the determination of a ship's place at sea; in Optics, they are employed to investigate the properties of light; and in Astronomy, to determine the distances of the heavenly bodies.

(4.) Algebra is that branch of mathematics which enables us, by means of letters and other symbols, to *abridge* and *generalise* the reasoning employed in the solution of all questions relating to numbers.

Arithmetic is the art or science of numbering. It treats of the nature and properties of numbers, but it is limited to certain methods of calculation which occur in common practice. Algebra is more comprehensive, and has been called by Newton, Universal Arithmetic.

(5.) The following are the main points of difference between Arithmetic and Algebra.

First, the operations of Algebra are more *general* than those of Arithmetic. In Arithmetic, we represent quantities by particular numbers, as 2, 5, 7, etc. which numbers always retain the same value. The results obtained, therefore, are applicable only to the particular question proposed. Thus, if it is required to find the interest of a thousand dollars for three months at six per cent. the question may be solved by Arithmetic, and we obtain an answer, which is applicable only to this problem.

But in the solution of an Algebraic problem, we employ letters to which any value may be attributed at pleasure. The results obtained, therefore, are equally applicable to all questions of a particular *class*. Thus, if we have given the sum and difference of two quantities, we may obtain by means of Algebra a general expression for the quantities themselves. This result will always be found true, whatever may be the magnitude of the quantities. Hence Algebra is adapted to the investigation of *general principles*; while Arithmetic is confined to operations upon *particular numbers*.

Secondly, Algebra enables us to solve a vast number of problems, which are *too difficult* for common Arithmetic. Some of the problems in Sections VII. and VIII. may be solved by Arithmetical methods; but others cannot be thus resolved, particularly such problems as are given in Sections XII. XIV. etc.

Thirdly, in Arithmetic all the different quantities which enter into a problem are *blended together* in the result, so as to leave no trace of the operations to which they have been subjected. From a simple inspection of the result, we cannot tell whether it was derived by multiplication or division, involution or evolution; or what connection it has with the given quantities of the problem. But in an Algebraic solution, all the different quantities are preserved *distinct* from each other, and we see at a glance, how all the data of the problem are combined in the result.

Fourthly, the operations of Algebra are often far more *concise* than those of Arithmetic. Thus, although some of the problems in Sections VII. and VIII. may be solved Arithmetically, these solutions are generally much more tedious than the Algebraic. This advantage which is possessed by Algebra is partly due to the representation of the unknown quantities by letters, and their introduction into the operations as if they were already known; and partly to the fact that the operations of multiplication, division, etc. are at first merely *indicated*, and are not actually performed until an Algebraic expression has been reduced to its simplest form.

Finally, perhaps the most striking difference between Arithmetic and Algebra, springs from the use of *negative quantities*, which give rise to many peculiar results.

The full purport of these remarks will be best apprehended, after the student has made some progress in the study of Algebra.

(6.) A *definition* is the explanation of any term or word. It is essential to a perfect definition that it distinguish the thing defined from every thing else. Thus, if we say that man is a biped, it is no definition of man, because there are many other bipeds.

(7.) A *theorem* is the statement of some property, the truth of which is required to be proved. Thus when it is said that the sum of the three angles of any triangle is equal to two right angles, this is a theorem, the truth of which is demonstrated by Geometry.

(8.) A *problem* is a question requiring something to be done. Thus, to draw one line perpendicular to another, is a problem. Theorems and problems are both known by the general term of *propositions.*

(9.) A *determinate* problem is one which admits of a certain or definite answer. An *indeterminate* problem commonly admits of an indefinite number of solutions; although when the answers are required in positive whole numbers, they are in some cases confined within certain limits, and in others the problem may be impossible.

(10.) The *solution* of a problem is the process by which we obtain the answer to it. A *numerical* solution is the obtaining an answer in numbers. A *geometrical* solution is the obtaining an answer by the principles of geometry. A *mechanical* solution is one which is gained by trials.

(11.) The principal symbols employed in Algebra are the following:

The sign + (an erect cross) is named *plus*, and is employed to denote the *addition* of two or more numbers. Thus, $5 + 3$ signifies that we must add 3 to the number 5, in which case the result is 8. In the same manner $11 + 6$ is equal to 17; $14 + 10$ is equal to 24, etc.

We also make use of the same sign to connect several numbers together. Thus $7 + 5 + 9$ signifies that to the number 7 we must add 5 and also 9 which makes 21.

So also the sum of $8 + 5 + 13 + 11 + 1 + 3 + 10$ is equal to 51.

(12.) In order to generalise numbers we represent them by letters, as a, b, c, d, etc. Thus the expression $a + b$ signifies the sum of two numbers, which we represent by a and b, and these may be any numbers whatever. In the same manner $m + n + b + x$ signifies the sum of the numbers represented by these four letters. If we knew, therefore, the numbers represented by the letters, we could easily find by arithmetic the value of such expressions.

The first letters of the alphabet are commonly used to represent *known* quantities, and the last letters those which are *unknown*.

(13.) The sign — (a horizontal line) is called *minus*, and indicates that one quantity is to be *subtracted* from another. Thus $8 - 5$, signifies that the number 5 is to be taken from the number 8, which leaves a remainder of 3. In like manner, $12 - 7$ is equal to 5; and $20 - 14$ is equal to 6, etc.

Sometimes we may have several numbers to subtract from a single one. Thus, $16 - 5 - 4$ signifies that 5 is to be subtracted from 16, and this remainder is to be further diminished by 4, leaving 7 for the result. In the same manner $50 - 1 - 3 - 5 - 7 - 9$ is equal to 25. So also, $a - b$ signifies that the number designated by a is to be diminished by the number designated by b.

Quantities preceded by the sign + are called *positive* quantities; those preceded by the sign —, *negative* quantities. When no sign is prefixed to a quantity, + is to be understood. Thus, $a + b - c$ is the same as $+ a + b - c$.

(14.) The sign × (an inclined cross) is employed to denote the *multiplication* of two or more numbers. Thus, 3×5 signifies that 3 is to be multiplied by 5, making 15. In like manner, $a \times b$ signifies a multiplied by b; and $a \times b \times c$ signifies the continued product of the numbers designated by a, b, and c; and so on for any number of quantities.

Multiplication is also frequently indicated by placing a point between the successive letters. Thus, $a . b . c . d$ signifies the same thing as $a \times b \times c \times d$.

Generally, however, when numbers are represented by letters, their multiplication is indicated by writing them in succession, without the interposition of any sign. Thus, $a\ b$ signifies the same thing as $a . b$ or, $a \times b$; and $a\ b\ c\ d$ is equivalent to $a . b . c . d$, or $a \times b \times c \times d$.

It must be remarked that the notation $a . b$ or $a\ b$ is seldom employed except when the numbers are designated by letters. If, for example, we attempt to represent the product of the numbers 5 and 6 in this manner; $5 . 6$ might be confounded with an integer followed by a decimal fraction; and 56 would be read *fifty-six*, according to the common system of notation.

The multiplication of numbers may, however, be expressed by placing a point between them, in cases where *no ambiguity* can arise from the use of this symbol. Thus, $1 . 2 . 3 . 4$ may represent the continued product of the numbers 1, 2, 3, 4.

(15.) When two or more quantities are multiplied together, each of them is called a *factor*. Thus, in the expression 7×5, 7 is a factor and so is 5. In the product abc there are three factors, a, b, c.

When a quantity is represented by a letter, it is called a *literal* factor, to distinguish it from a *numerical* factor which is represented by an Arabic numeral. Thus, in the expression $5ab$, 5 is a numerical factor, while a and b are literal factors.

(16.) The character $\div$ (a horizontal line with a point above and below) shows that the quantity which precedes it, is to be divided by that which follows.

Thus, $24 \div 6$ signifies that 24 is to be divided by 6, making 4. So also $a \div b$ is a divided by b.

Generally, however, the division of two numbers is indicated by writing the dividend above the divisor, and drawing a line between them.

Thus $24 \div 6$ and $a \div b$ are usually written $\frac{24}{6}$ and $\frac{a}{b}$.

(17.) The sign $=$ (two horizontal lines) when placed between two quantities denotes that they are equal to each other.

Thus $7 + 6 = 13$, signifies that the sum of 7 and 6 is equal to 13. So also \$1 = 100 cents, is read one dollar equals one hundred cents. In like manner $a = b$, signifies that a is equal to b; and $a + b = c - d$, signifies that the sum of the numbers designated by a and b is equal to the difference of the numbers designated by c and d.

(18.) The symbol $>$ is called the sign of *inequality*, and when placed between two numbers, denotes that one of them is *greater* than the other, the opening of the sign being turned towards the greater number.

Thus $3<5$, signifies that 3 is less than 5; and $11>6$ denotes that 11 is greater than 6. So also $a>b$ shows that a is greater than b; and $c<d$ shows that c is less than d.

(19.) A *coefficient* is employed to denote that a quantity is *added* to itself a certain number of times.

Thus, instead of writing $a+a+a+a+a$, which represents five a's added together, we write $5a$, where 5 is the coefficient of a. In like manner $10\,ab$ signifies ten times the product of a and b. The coefficient may be either a whole number or a fraction. Thus, $\frac{3}{4}\,a$ signifies three-fourths of a. When no coefficient is expressed, 1 is always to be understood. Thus, $1\,a$ and a signify the same thing.

The coefficient may be a *letter* as well as a figure. In the expression mn, m may be considered as the coefficient of n, because n is to be taken as many times as there are units in m. If m stands for 5, then mn is 5 times n.

In $4abc$, 4 may be considered as the coefficient of abc; or $4a$ may be considered as the coefficient of bc; or $4ab$ the coefficient of c.

(20.) An *exponent* or *index* is employed to denote that a quantity is *multiplied* by itself a certain number of times.

Thus, instead of writing $a\times a\times a\times a\times a$, or $aaaaa$, which represents five a's multiplied together, we write a^5, where 5 is called the exponent or index of a. So also, $b\times b\times b\times b\times b\times b$, or $b.b.b.b.b.b$, or $bbbbbb$, or the continued product of six b's, is written more briefly b^6, where 6 is the exponent of b.

When no exponent is expressed, 1 is always understood. Thus, a^1 and a signify the same thing.

Exponents may be attached to figures as well as letters.

Thus, the product of	3 by 3	may be written	3^2,	which equals	9
"	$3\times3\times3$	"	3^3,	"	27
"	$3\times3\times3\times3$	"	3^4,	"	81
"	$3\times3\times3\times3\times3$	"	3^5,	"	243

(21.) The products formed by the successive multiplication of the same number by itself, are called the *powers* of that number.

Thus, let a represent any number, its *first power* is simply a, or a^1.

Its *second power*, or square is aa, or a^2.

Its *third power*, or cube is aaa, or a^3.

In general, any power of a quantity, is designated by the number of factors which form the product; and the exponent denotes the *degree* of the power. Thus, $3 \times 3 \times 3 \times 3 \times 3$, or 3^5 which equals 243, is the fifth power of 3; and the power is of the fifth degree.

(22.) A *root* of a quantity is that which multiplied by itself a certain number of times will produce the given quantity.

The symbol $\sqrt{}$ is called the *radical sign*, and when prefixed to a quantity denotes that its root is to be extracted. Thus,

$\sqrt[2]{9}$, or simply $\sqrt{9}$, denotes the square root of 9, which is 3.

$\sqrt[3]{64}$ denotes the cube root of 64, which is 4.

$\sqrt[4]{16}$ denotes the fourth root of 16, which is 2.

So also,

$\sqrt[2]{a}$, or simply $\sqrt{a}$, is the square root of a.

$\sqrt[3]{a}$ denotes the third or cube root of a.

$\sqrt[4]{a}$ denotes the fourth root of a.

$\sqrt[n]{a}$ denotes the nth root of a, where n may represent any number whatever.

The number placed over the radical sign, is called the *index* of the root. Thus, 2 is the index of the square root, 3 of the cube root, 4 of the fourth root, and n of the nth root. The index of the square root is usually omitted. Thus, instead of $\sqrt[2]{ab}$, we usually write $\sqrt{ab}$.

(23.) When four quantities are *proportional*, the proportion is expressed by points as in arithmetic. Thus, $a : b :: c : d$, signifies that a has to b the same ratio which c has to d.

(24.) A *vinculum* ——, or a *parenthesis* ( ), indicates that several quantities are to be subjected to the *same operation.*

Thus, $\overline{a+b+c} \times d$, or $(a+b+c) \times d$, denotes that the sum of a, b and c is to be multiplied by d. But $a+b+c \times d$, denotes that c *only* is to be multiplied by d.

When the parenthesis is used, the sign of multiplication is generally omitted. Thus, $(a+b+c) \times d$, is the same as $(a+b+c)\,d$, or $d\,(a+b+c)$.

(25.) Three dots $\because$ are sometimes employed to denote *therefore* or *consequently.*

There are a few other symbols employed in algebra, in addition to those already enumerated, which will be explained as they occur.

(26.) Every number written in algebraic language, that is by aid of algebraic symbols, is called an *algebraic quantity*, or an *algebraic expression.*

Thus $3a$, is the algebraic expression for three times the number a.

$4a^2$ is the algebraic expression for four times the square of the number a.

$7a^3b^4$ is the algebraic expression for seven times the third power of a multiplied by the fourth power of b.

(27.) An algebraic quantity not combined with any other by the sign of addition or subtraction, is called a *monomial*, or a quantity of one term, or simply a *term.*

Thus, $2a$, $5bc$, and $7xy^2$ are monomials.

(28.) An algebraic expression, which is composed of several terms, is called a *polynomial.*

Thus, $a+2b+5c-d$ is a polynomial.

A polynomial consisting of *two terms* only, is usually called a *binomial;* when consisting of *three terms*, a *trinomial.*

Thus, $3a+5b$ is a binomial; and $a+3cb+xy$ is a trinomial.

(29.) The *numerical value* of an algebraic expression is the result obtained when we attribute particular values to the letters.

Let the expression be $2a^2b$.

If we suppose $a=2$ and $b=3$, the value of this expression will be $2\times2\times2\times3=24$.

If we suppose $a=4$, and $b=3$, the value of the same expression will be $2\times4\times4\times3=96$.

The numerical value of a polynomial is not affected by *changing the order* of the terms, provided we preserve their respective signs.

The expressions $a^2+2ab+b^2$, a^2+b^2+2ab, $b^2+2ab+a^2$ have all the same numerical value.

Thus if $a=5$ and $b=2$, the value of a^2 will be 25, that of $2ab$ will be 20, and b^2 will be 4; and if these numbers be added together, their sum will be the same in whatever order they are placed. Thus,

$$\begin{array}{rrrrrr} 25 & 25 & 20 & 20 & 4 & 4 \\ 20 & 4 & 25 & 4 & 25 & 20 \\ 4 & 20 & 4 & 25 & 20 & 25 \\ \hline 49 & 49 & 49 & 49 & 49 & 49 \end{array}$$

(30.) Each of the literal factors which compose a term is called a *dimension* of this term; and the *degree* of a term is the *number* of these factors or dimensions. A numerical coefficient is not counted as a dimension.

Thus, $3a$ is a term of one dimension, or of the first degree.

$5ab$ is a term of two dimensions, or of the second degree.

$6a^2bc^3$ is a term of six dimensions, or of the sixth degree.

In general, the degree, or the number of dimensions of a term is equal to the sum of the exponents of the letters contained in the term.

Thus, the degree of the term $5ab^2cd^3$ is $1+2+1+3$ or 7; that is, this term is of the seventh degree.

(31.) A polynomial is said to be *homogeneous*, when all its terms are of the *same degree*.

Thus, $2a-3b+c$ is of the first degree and homogeneous.

$3a^2-4ab+b^2$ is of the second degree and homogeneous.

$2a^3+3a^2c-4c^2d$ is of the third degree and homogeneous.

$5a^3-2ab+c$ is *not* homogeneous.

(32.) *Like* or *similar* terms are terms composed of the *same letters* affected with the *same exponents*.

Thus, $3ab$ and $7ab$ are similar terms.

$5a^2c$ and $3a^2c$ are also similar terms.

But $3ab^2$ and $4a^2b$ are *not* similar, for although they contain the same letters, the same letters are not affected with the same exponents.

(33.) The *reciprocal* of a quantity is the quotient arising from dividing a unit by that quantity.

Thus, the reciprocal of 2 is $\frac{1}{2}$; the reciprocal of a is $\frac{1}{a}$

(34.) A few examples are here subjoined, to exercise the learner on the preceding definitions and remarks.

Examples in which words are to be converted into algebraic symbols.

1. What is the algebraic expression for the following statement? The second power of a, increased by twice the product of a and b, diminished by c, and increased by d, is equal to seventeen times f.

Ans. $a^2+2ab-c+d=17f$.

2. The quotient of three divided by the sum of x and four, is equal to twice b diminished by eight.

3. One third of the difference between six times x and four, is equal to the quotient of five divided by the sum of a and b.

4. Three quarters of x increased by five, is equal to three sevenths of b diminished by seventeen.

5. One ninth of the sum of six times x and five, added to one third of the sum of twice x and four, is equal to the product of a, b and c.

6. The quotient arising from dividing the sum of a and b by the product of c and d, is equal to four times the sum of e, f, g, and h.

(35.) *Examples in which the algebraic signs are to be translated into common language.*

1. $\frac{x+a}{b}+\frac{x}{c}=\frac{d}{a+b}$.

Ans. The quotient arising from dividing the sum of x and a by b, increased by the quotient of x divided by c, is equal to the quotient of d divided by the sum of a and b.

2. $7a^2 + (b - c) \times (d + e) = g + h$

3. $\dfrac{a+g}{3+b-c} + \dfrac{6-4m}{3} = \dfrac{h}{7+a}$

4. $4\sqrt{ab} - 25 = \dfrac{2b+c}{3a-d}$

5. $2a\sqrt{b^2 - ac} = 5(h + d + x)$

6. $\dfrac{\sqrt{5b} + 3\sqrt{c}}{2a+1} = 5x + \frac{1}{4}$

(36.) Find the value of the following expressions, when $a = 6$, $b = 5$ and $c = 4$.

1. $a^2 + 3ab - c^2$.

Ans. $36 + 90 - 16 = 110$.

2. $a^2 \times (a + b) - 2abc$

3. $\dfrac{a^3}{a+3c} + c^2$

4. $c + \dfrac{2bc}{\sqrt{2ac + c^2}}$

5. $\sqrt{b^2 - ac} + \sqrt{2ac + c^2}$

6. $3\sqrt{c} + 2a\sqrt{2a + b + 2c}$

SECTION II.

ADDITION.

(37.) Addition is the connecting of quantities together by means of their proper signs, and incorporating such as can be united into one sum.

It is convenient to distinguish three cases.

CASE I.

When the quantities are similar and have the same signs.

RULE.

Add the coefficients of the several quantities together, and to their sum annex the common letter or letters, prefixing the common sign.

Thus, the sum of $3a$ and $5a$ is obviously $8a$. So also, $-3a$ and $-5a$ make $-8a$; for the minus sign before each of the terms shows that they are to be subtracted, not from each other, but from some quantity which is not here expressed; and if $3a$ and $5a$ are to be successively subtracted from the same quantity, it is the same as subtracting at once $8a$.

EXAMPLES.

$$\begin{array}{ccccc}
3a & -3ab & 2b+3x & a-2x^2 & 2a+y^2 \\
5a & -6ab & 5b+7x & 4a-3x^2 & 5a+2y^2 \\
7a & -ab & b+2x & 3a-5x^2 & 9a+3y^2 \\
a & -7ab & 4b+3x & 7a-x^2 & 4a+6y^2 \\
\hline
16a & & & &
\end{array}$$

The learner must continually bear in mind the remark of Art. 13, that when no sign is prefixed to a quantity, plus is always to be understood.

CASE II.

(38.) When the quantities are similar, but have different signs.

RULE.

Add all the positive coefficients together, and also all those that are negative; subtract the least of these results from the greater; to the difference annex the common letter or letters, and prefix the sign of the greater sum.

Thus, instead of $7a - 4a$, we may write $3a$; since these two expressions obviously have the same value.

Also, if we have $5a - 2a + 3a - a$, this signifies that from $5a$ we are to subtract $2a$, add $3a$ to the remainder, and then subtract a from this last sum; the result of which operation is $5a$. But it is generally most convenient to take the sum of the positive quantities, which in this case is $8a$; then take the sum of the negative quantities, which in this case is $3a$; and we have $8a - 3a$ or $5a$, the same result as before.

EXAMPLES.

$-3a$	$6x + 5ay$	$2ay - 7$	$-2a\sqrt{x}$	$-6a^2 + 2b$
$+7a$	$-3x + 2ay$	$-ay + 8$	$a\sqrt{x}$	$2a^2 - 3b$
$+8a$	$x - 6ay$	$2ay - 9$	$-3a\sqrt{x}$	$-5a^2 - 8b$
$-a$	$2x + ay -$	$3ay - 11$	$7a\sqrt{x}$	$4a^2 - 2b$
$+11a$				

CASE III.

(39.) When some of the quantities are dissimilar.

RULE.

Collect all the like quantities together, by taking their

sums or differences as in the two former cases, and set down those that are unlike, one after another, with their proper signs.

Unlike quantities cannot be united in one term. Thus, $2a$ and $3b$ neither make $5a$ nor $5b$. Their sum can only be written $2a + 3c$.

EXAMPLES.

$$\begin{array}{rrrr}
2xy - 2x^2 & 3x^2y \quad 2a\sqrt{x} & 2ax - \quad 220 & 2\sqrt{x} - 18y \\
3x^2 + xy & -2xy^2 - ax^2 & 3x^2 - \quad 2ax & 3\sqrt{xy} + 10x \\
x^2 - xy & -3y^2x \quad 3ax^2 & 5x^2 - 13\sqrt{x} & 2x^2y + 25y \\
4x^2 - 3xy & -8x^2y - \sqrt{x} & 3\sqrt{x} + 100 & 12xy - \sqrt{xy} \\
\hline
6x^2 - xy & & &
\end{array}$$

(40.) When several quantities are to be added together, it is most convenient to *write all the similar terms under each other*, as in the following example.

1. Add together

$$\begin{array}{l}
11bc + 4ad - 8ac + 5cd \\
8ac + 7bc - 2ad + 4mn \\
2cd - 3ab + 5ac + an \\
9an - 2bc - 2ad + 5cd
\end{array}$$

These terms may be written thus:

$$\begin{array}{llllll}
11bc + 4ad - 8ac + 5cd + an + 4mn - 3ab \\
7bc - 2ad + 8ac + 2cd + 9an \\
-2bc - 2ad + 5ac + 5cd \\
\hline
\text{Sum} \quad 16bc \qquad + 5ac + 12cd + 10an + 4mn - 3ab.
\end{array}$$

2. Add together the quantities,

$$\begin{array}{l}
7m + 3n - 14p + 17n \\
3a + 9n - 11m + 2r \\
5p - 4m + 8n \\
11n - 2b - m - r + s
\end{array}$$

3. Add together

$4a^2b + 3c^3d - 9m^2n$
$4m^2n - ab^2 + 5c^3d + 7a^2b$
$6m^2n - 5c^3d + 4mn^2 - 8ab^2$
$7mn^2 + 6c^3d - 5m^2n - 6a^2b$
$7c^3d - 10ab - 8m^2n - 10d^4$
$12a^2b - 6ab^2 + 2c^3d + mn$

4. Add together

$3b - a - c - 115d$
$6c - 5f - d$
$3a - 2b - 3c + 27e$
$3e - 7f + 5b - 8c$
$17c - 6b - 7a$
$11f - 5e + 9d$
$g - 3a - 5c$
$6e - 2d - 9f$

5. Add together

$2ab^3 + 3ac^2 - 8cx^2 + 9b^2x - 8hy^2 + 10ky$
$5a^2 - 4ab^2 - 7bx^2 - b^2x - 4ky^2 - 15hy$
$5ky - hy^2 + 11x + 14b^3 - 22ac - 10x^2$
$19ac^2 - 8b^2x + 9x^2 + 6hy + 2ky^2 + 2ab^2$

(41.) It must be observed that the term addition is used in a more extended sense in algebra than in arithmetic. In arithmetic, where all quantities are regarded as positive, addition implies *augmentation*. The sum of two quantities will therefore, be numerically greater than either quantity. Thus, the sum of 7 and 5 is 12, which is numerically greater than either 5 or 7.

But in Algebra we consider negative as well as positive quantities; and by the sum of two quantities, we mean their aggregate, regard being paid to their signs. Thus, the sum of $+7$ and -5 is $+2$, which is numerically less than either 7 or 5. So also, the sum of $+a$ and $-b$ is $a-b$. In this case, the algebraic sum is numerically the *difference* of the two quantities.

This is one instance among many, in which the same terms are used in a much more general sense in the higher mathematics than they are in arithmetic.

SECTION III.

SUBTRACTION.

(42.) Subtraction is the taking of one quantity from another; or it is finding the difference between two quantities or sets of quantities.

Let it be required to subtract $8 - 3$ from 15.

Now $8 - 3$ is equal to 5.

And 5 subtracted from 15 leaves 10.

The result, then, must be 10. But to perform the operation on the numbers as they were given, we first subtract 8 from 15 and obtain 7. This result is *too small* by 3, because the number 8 is *larger by* 3 than the number which was required to be subtracted. Therefore, in order to correct this result, the 3 must be added, and we have

$$15 - 8 + 3 = 10, \text{ as before.}$$

Again, let it be required to subtract $5a - 3b$ from $8a + 5b$. It is plain that if the part $5a$ were alone to be subtracted, the remainder would then be

$$8a + 5b - 5a.$$

But as the quantity actually proposed to be subtracted, is *less* than $5a$ by $3b$, *too much* has been taken away by $3b$; and therefore, the true remainder will be greater than $8a + 5b - 5a$ by $3b$, and will hence be expressed by

$$8a + 5b - 5a + 3b,$$

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where the signs of the last two terms are both *contrary* to what they were given in the subtrahend.

(43.) Hence we deduce the following general

RULE.

Conceive the signs of all the terms which are to be subtracted, to be changed from + to —, or from — to +, and then collect the terms together as in the several cases of addition.

It is better in practice to leave the signs of the subtrahend *unchanged*, and simply *conceive* them to be changed, that is, treat the quantities as if the signs were changed; for otherwise, when we come to revise the work to detect any error in the operation, we might often be in doubt as to what were the signs of the quantities as originally proposed.

EXAMPLES.

From	$5a^2-2b$	$5xy+8x-2$	$10-8x-3xy$	$4\sqrt{ax}-2x^2y$
Subtract	$2a^2+5b$	$3xy-8x-7$	$-x+3\quad -xy$	$3\sqrt{ax}-5xy^2$
Remainder	$3a^2-7b$			

From $5a+4b-2c+7d$
Take $3a+2b+\ \ c+5d$

From $11xy+2y^2-16x^2$
Take $-4xy+6y^2-18x^2$

From $6aby-4yx+4zx$
Take $-3aby+5zx+3yx$

From $x^2+2xy+y^2$
Take $x^2-2xy+y^2$

From $3a^2+\ \ ax+2x^2-14a^2x+19ax^2-4x^3\quad\ \ +5a^2x^2$.
Take $2a^2-4ax+x^2\ \ -15a^2x+11ax^2-15a^2x^2-4x^3$

Subtraction may be *proved* as in Arithmetic, by adding the remainder to the subtrahend. The sum should be equal to the minuend.

(44.) The term subtraction, it will be perceived, is used in a more general sense in algebra than in arithmetic. In arithmetic, where all quantities are regarded as positive, a number is always *diminished* by subtraction. But in algebra, the difference between two quantities may be numerically greater than either. Thus, the difference between $+a$ and $-b$ is $a+b$.

The distinction between positive and negative quantities may be illustrated by the degrees upon a thermometer. The degrees above zero are considered positive, and those below zero negative. From five degrees above zero, to five degrees below zero, the numbers stand thus:

$$+5, +4, +3, +2, +1, 0, -1, -2, -3, -4, -5.$$

The difference between five degrees above zero and five degrees below zero, is ten degrees, which is numerically the *sum* of the two quantities.

(45.) In practice, it is often sufficient merely to *indicate* the subtraction of a polynomial, without actually performing the operation. This is done by enclosing the polynomial in a parenthesis, and prefixing the sign —.

$$\text{Thus, } 5a-3b+4c-(3a-2b+8c)$$

signifies that the quantity $3a-2b+8c$ is to be subtracted from $5a-3b+4c$. The subtraction is here merely *indicated*. If we actually perform the operation, the expression becomes

$$5a-3b+4c-3a+2b-8c$$
$$\text{or } 2a-b-4c.$$

(46.) According to the preceding principle, polynomials may be written in a variety of forms.

$$\text{Thus } a-b-c+d$$
$$\text{is equivalent to } a-(b+c-d)$$
$$\text{or to } a-b-(c-d)$$
$$\text{or to } a+d-(b+c).$$

Transformations of this sort, which consist in decomposing a polynomial into two parts separated from each other by th sign —, are of frequent use in algebra. It is recommended to the student to write out polynomials like the above, containing both positive and negative terms, in all the possible modes, including several terms in a parenthesis.

In the following examples, let the results all be reduced to their simplest form.

1. $a+b-(2a-3b)-(5a+7b)-(-13a+2b)=$.
2. $37a-5f-(3a-2b-5c)-(6a-4b+3h)=$.
3. $8a^2xy-5bx^2y+17cxy^2-9y^5-(a^2xy+3bx^2y-13cxy^2+20y^5)=$.
4. $28ax^2-16a^2x^2+25a^3x-13a^4-(18ax^2+20a^2x^2-24a^3x-7a^4)=$

(47.) It has already been remarked that algebra differs from arithmetic in the use of negative quantities, and it is important that the beginner should obtain clear ideas of their nature.

In many cases, the terms positive and negative are merely *relative.* They indicate some sort of *opposition* between two classes of quantities, such that if one class should be *added*, the other ought to be *subtracted.* Thus, if a ship sails alternately northward and southward, and the motion in one direction is called *positive*, the motion in the opposite direction should be considered *negative.*

Suppose a ship, setting out from the equator, sails northward 50 miles, then southward 27 miles, then northward 15 miles, then southward again 22 miles, and we wish to determine the last position of the ship. Calling the northerly motion +, the whole may be expressed algebraically, thus:

$$+50-27+15-22$$

which reduced, equals +16. The positive sign of the result, indicates that the ship was 16 miles *north* of the equator.

Suppose the same ship sails again 8 miles north, then 35 miles south, the whole may be expressed thus:

$$+50-27+15-22+8-35$$

which reduces to —11. The negative sign of the result indicates that the ship was now 11 miles *south* of the equator.

In this example, we have considered the northerly motion +, and the southerly motion —; but we might with equal propriety have considered the southerly motion +, and the northerly motion —. It is, however, indispensible, that we adhere to the *same system* throughout, and retain the proper sign of the result, as this sign shows whether the ship was at any time north or south of the equator.

In the same manner, if we consider easterly motion +, westerly motion must be regarded as —, and vice versa. And generally, when quantities which are estimated in different directions enter into the same algebraic expression, those which are measured in one direction being treated as +, those which are measured in the opposite direction must be regarded as —.

So also in estimating a man's property, *gains* and *losses* being of an opposite character, must be affected with different signs. Suppose a man with a property of 1000 dollars, loses 300 dollars, afterwards gains 100, and then loses again 400 dollars. The whole may be expressed algebraically, thus:

$$+1000-300+100-400$$

which reduced, equals $+400$. The + sign of the result indicates that he has now 400 dollars remaining in his possession. Suppose he further gains 50 dollars and then loses 700 dollars. The whole may now be expressed thus:

$$+1000-300+100-400+50-700$$

which reduces to -250. The — sign of the result indicates that his losses exceed the sum of all his gains, and the property originally in his possession; in other words, he owes 250 dollars more than he can pay, or in common language he is 250 dollars *worse than nothing*.

This phraseology must not be regarded as wholly figurative. For in algebra, a negative quantity standing alone, is regarded as less than nothing; and of two negative quantities, that which

is *numerically the greatest*, is considered as the *least*. For if from the same number, we subtract successively numbers larger and larger, the remainders must continually *diminish*. Take any number, 5 for example, and from it subtract successively 1, 2, 3, 4, 5, 6, 7, 8, 9, etc. we obtain
$5-1$, $5-2$, $5-3$, $5-4$, $5-5$, $5-6$, $5-7$, $5-8$, $5-9$, etc. or reducing

$$4, 3, 2, 1, 0, -1, -2, -3 -4.$$

Whence we see that -1 should be regarded as smaller than nothing; -2 less than -1; -3 less than -2, etc.

SECTION IV.

MULTIPLICATION.

(48.) Multiplication is repeating the multiplicand, as many times as there are units in the multiplier.

When several quantities are to be multiplied together, the result will be the same, in whatever *order* the multiplication is performed.

Thus 2 × 3 × 4 is equal to 2 × 4 × 3 or 4 × 3 × 2 or 3 × 4 × 2, the product in each case being 24. So also, if a, b and c represent any three numbers, we shall have abc equal to bca or cab.

It is convenient to consider the subject of multiplication under several cases.

CASE I.

(49.) *When the factors are both monomials.*

From Article 14 it appears that in order to represent the multiplication of two monomials, such as $3abc$ and $5def$, we may write these quantities in succession without interposing any sign, and we shall have

$$3abc5def$$

But according to the principle stated in the preceding Article, this result may be written

$$3 \times 5abcdef, \text{ or } 15abcdef.$$

Hence we deduce the following

RULE.

Multiply the coefficients of the two terms together, and to the product annex all the different letters in succession.

EXAMPLES.

Multiply	$12a$	$5a$	$7ab$	$7axy$	$6xyz$
By	$3b$	$6x$	$5ac$	$6ay$	ayz
Product	$36ab$				

From Article 48, it appears to be immaterial in what *order* the letters of a term are arranged; it is, however, generally most convenient to arrange them *alphabetically*.

(50.) We have seen in Art. 20, that when the same letter appears several times as a factor in a product, this is briefly expressed by means of an exponent. Thus aaa is written a^3; the number 3 showing that a enters three times as a factor. Hence, if the same letters are found in two monomials which are to be multiplied together, the expression for the product may be abbreviated by adding the exponents of the same letters. Thus, if we are to multiply a^3 by a^2, we find a^3 equivalent to aaa, and a^2 to aa. Therefore the product will be $aaaaa$, which may be written a^5, a result which we might have obtained at once by adding together 3 and 2, the exponents of the common letter a.

Hence, since every factor of both multiplier and multiplicand must appear in the product, we have the following

RULE FOR THE EXPONENTS.

Powers of the same quantity may be multiplied by adding their exponents.

EXAMPLES.

Multiply	$8a^2bc^2$	$2a^3b^2c$	$5a^4b^3c^2$	$2a^2b^3c^4$
By	$7abcd^2$	$8abc^3$	$7a^3b^4c^3d$	$5a^5bc^3$
Product	$56a^3b^2c^3d^2$			

CASE II.

(51.) *When the multiplicand is a polynomial.*

If $a+b$ is to be multiplied by c, this implies that the sum of the units in a and b is to be repeated c times; that is, the units in b repeated c times must be added to the units in a repeated also c times. Hence we deduce the following

RULE.

Multiply each term of the multiplicand separately by the multiplier, and add together the products.

EXAMPLES.

Multiply	$3a+2b$	a^2+2x+1	$3y^2+5xy+2$	$3x^3+xy+2y^2$
By	$4a$	$4x$	xy	$5x^2y$
Product	$12a^2+8ab$			

CASE III.

(52.) *When both the factors are polynomials.*

If $a+b$ is to be multiplied by $c+d$, this implies that the quantity $a+b$ is to be repeated as many times as there are units in the sum of c and d; that is, we are to multiply $a+b$ by c and d successively, and add the partial products. Hence we deduce the following

RULE.

Multiply each term of the multiplicand by each term of the multiplier separately, and add together the products.

EXAMPLES.

Multiply	$a+b$	$3x+2y$	$ax+b$	$3a+x$
By	$a+b$	$2x+3y$	$cx+d$	$2a+4x$
Product	$a^2+2ab+b^2$			

When several terms in the product are *similar*, it is most convenient to set them under each other, and then unite them by the rules for addition.

(53.) The examples thus far given in multiplication have been confined to *positive* quantities, and the products have all been positive. We must now establish a general rule for the signs of the product.

First, if $+a$ is to be multiplied by $+b$, this signifies that $+a$ is to be repeated as many times as there are units in b, and the result is $+ab$. That is, a plus quantity multiplied by a plus quantity, gives a plus result.

Secondly, if $-a$ is to be multiplied by $+b$, this signifies that $-a$ is to be repeated as many times as there are units in b. Now $-a$ taken twice is obviously $-2a$, taken three times is $-3a$, etc. hence if $-a$ is repeated b times, it will make $-ba$ or $-ab$. That is, a minus quantity multiplied by a plus quantity gives minus.

Thirdly, if $+a$ is to be multiplied by $-b$, we may resolve it into the preceding case, for 3 multiplied by 2 is equal to 2 multiplied by 3. The quantity a repeated b times, is equal to the quantity b repeated a times. Hence $+a$ multiplied by $-b$ is equal to $-ab$; that is, a plus quantity multiplied by a minus quantity gives minus.

Fourthly, to determine the product of a minus quantity by a minus quantity, let it be proposed to multiply $a-b$ by $c-d$. By this we understand that the quantity $a-b$ is to be repeated as many times as there are units in $c-d$. If we multiply $a-b$ by c we obtain $ac-bc$. But $a-b$ was only to be taken $c-d$ times; therefore in this first operation we have repeated it too many times by the quantity d. Hence, to have the true product, we must subtract d times $a-b$ from $ac-bc$. But d times $a-b$ is equal to $ad-bd$, which subtracted from $ac-bc$ gives

$$ac-bc-ad+bd.$$

Thus we see that $-b$ multiplied by $-d$ gives $+bd$; that is a minus quantity multiplied by a minus quantity gives plus. This demonstration obviously comprehends the preceding case, since we find that $+a$ multiplied by $-d$ gives $-ad$.

Let it be required to multiply $8-5$ by $6-2$.

If we multiply $8-5$ by 6, we obtain $48-30$; that is, we have repeated $8-5$ six times. But it was only required to repeat the multiplicand four times, or $(6-2)$. We must therefore diminish this product by twice $(8-5)$ which is $16-10$; and this subtraction is performed by changing the signs of the subtrahend; hence we have

$$48-30-16+10$$

which is equal to 12. This result is obviously correct, for $8-5$ is equal to 3, and $6-2$ is equal to 4; that is, it was required to multiply 3 by 4, the result of which is 12 as found above.

(54.) The preceding results may be briefly expressed, as follows:

$+$ multiplied by $+$, and $-$ multiplied by $-$ give $+$.

$+$ multiplied by $-$, and $-$ multiplied by $+$ give $-$.

Or, the product of two quantities having the *same sign*, has the sign *plus;* the product of two quantities having *different signs*, has the sign *minus:*

(55.) The following rule then comprehends the whole doctrine of multiplication.

Multiply each term of the multiplicand, by each term of the multiplier, and add together all the partial products, observing that like signs require + in the product, and unlike signs —.

EXAMPLE I.

$$\begin{array}{ll}
\text{Multiply} & 5a^4-2a^3b+4a^2b^2 \\
\text{By} & \quad a^3-4a^2b+2b^3 \\
\hline
\text{Partial Products} \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & \begin{array}{l} 5a^7-2a^6b+4a^5b^2 \\ \qquad -20a^6b+8a^5b^2-16a^4b^3 \\ \qquad\qquad\qquad\qquad +10a^4b^3-4a^3b^4+8a^2b^5 \end{array} \\
\hline
\text{Result} & 5a^7-22a^6b+12a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5
\end{array}$$

Ex. 2. Multiply $4a^3 - 5a^2b - 8ab^2 + 2b^3$ by $2a^2 - 3ab - 4b^2$

Ex. 3. Multiply $3a^2 - 5bd + ef$ by $-5a^2 + 4bd - 8ef$

Ex. 4. Multiply $x^4 + 2x^3 + 3x^2 + 2x + 1$ by $x^2 - 2x + 1$

Ex. 5. Multiply $14a^3e - 6a^2bc + c^2$ by $14a^3e + 6a^2bc - c^2$

Ex. 6. Multiply $3a^3 + 35a^2b - 17ab^2 - 13b^3$ by $3a^2 + 26ab - 57b^2$.

(56.) Since in the multiplication of two monomials, every factor of both quantities appears in the product, it is obvious that the *degree* of the product will be equal to the *sum* of the degrees of the multiplier and multiplicand. Hence also, if two polynomials are *homogeneous*, their product will be homogeneous.

Thus in the first of the preceding examples, all the terms of the multiplicand being of the fourth degree, and those of the multiplier of the third degree, all the terms of the product are of the seventh degree. For a like reason, in the second example, all the terms of the product are of the fifth degree; in the third example, they are of the fourth degree; and in the sixth example, they are of the fifth degree.

This remark will enable us to detect any error in the multiplication, so far as concerns the exponents. For example, if we find in one of the terms of a product which should be homogeneous, the sum of the exponents equal to 6, while in all the other terms it is equal to 7, a mistake has evidently been committed in the formation of one of the terms.

(57.) When the product arising from the multiplication of two polynomials does not admit of any reduction of similar terms, the *whole number* of terms in the product is equal to the number of terms in the multiplicand, multiplied by the number of terms in the multiplier.

Thus if we have five terms in the multiplicand, and four terms in the multiplier, the whole number of terms in the product will be 5 × 4, or 20. In general, if there be m terms in the multiplicand, and n terms in the multiplier, the whole number of terms in the product will be $m \times n$.

(58.) If the product contains *similar terms*, the number of terms in the product when reduced may be much less; but it is important to observe that among the different terms of the product, there are some which *cannot be reduced* with any others. These are:

1. The term arising from multiplying that term of the multiplicand, which is affected with the *highest* exponent of any one of the letters, by that term of the multiplier which is affected with the highest exponent of the same letter.

2. The term arising from the multiplication of the two terms affected with the *lowest* exponent of the same letter.

For it is evident from the rule of exponents, that these two partial products must involve the letter in question, the one with a *higher*, and the other with a *lower* exponent than any of the other partial products, and therefore cannot be similar to any of them. This remark is the foundation of the rule for division, in Art. 74.

(59.) For many purposes, it is sufficient merely to *indicate* the multiplication of two polynomials, without actually performing the operation. This is effected by enclosing the quantities in parentheses, and writing them in succession with or without the interposition of any sign.

Thus $(a + b + c)\,(d + e + f)$ signifies that the sum of a, b and c is to be multiplied by the sum of d, e and f.

When the multiplication is actually performed, the expression is said to be *expanded*.

(60.) The following Theorems are of such extensive application that they should be carefully committed to memory.

THEOREM I.

The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.

$$\begin{array}{ll}
\text{Thus if we multiply} & a + b \\
\text{By} & a + b \\
\hline
& a^2 + ab \\
& \qquad ab + b^2 \\
\hline
\text{We obtain the product} & a^2 + 2ab + b^2.
\end{array}$$

Thus if we wish to obtain the square of a binomial, we can write out the terms of the result at once according to this theorem without the necessity of performing an actual multiplication.

EXAMPLES.

1. $(x+y)^2=$
2. $(2a+b)^2=$
3. $(a+2b)^2=$
4. $(a+3b)^2=$
5. $(3a+3b)^2=$
6. $(4a+3b)^2=$
7. $(5a^2+b)^2=$
8. $(5a^2+7ab)^2=$
9. $(5a^3+b)^2=$
10. $(5a^3+8a^2b)^2=$

This theorem deserves particular attention, for one of the most common mistakes of beginners is to call the square of $a+b$ equal to a^2+b^2.

THEOREM II.

(61.) *The square of the difference of two quantities, is equal to the square of the first, minus twice the product of the first and second, plus the square of the second.*

Thus if we multiply $\quad a-b$

By $\quad a-b$

$$\begin{array}{l} \hline a^2-ab \\ \quad -ab+b^2 \\ \hline \end{array}$$

We obtain the product $\quad a^2-2ab+b^2$

EXAMPLES.

1. $(x-y)^2=$
2. $(a-2b)^2=$
3. $(2a-2b)^2=$
4. $(2a-3b)^2=$
5. $(5a-4b)^2=$
6. $(6a^2-x)^2=$
7. $(6a^2-3x)^2=$
8. $(7a^2-b)^2=$
9. $(7a^2-12ab)^2=$
10. $(7a^2b^2-12ab)^2=$

Here also, beginners often commit the mistake of putting the square of $a - b$ equal to $a^2 - b^2$.

THEOREM III.

(62.) *The product of the sum and difference of two quantities is equal to the difference of their squares.*

Thus if we multiply $a + b$

By $a - b$

$$\begin{array}{ll} a^2 + ab & \\ \quad - ab & - b^2 \\ \hline a^2 & - b^2 \end{array}$$

We obtain the product $a^2 - b^2$

EXAMPLES.

1. $(x + y)(x - y) =$
2. $(2a + b)(2a - b) =$
3. $(3a + b)(3a - b) =$
4. $(3a + 4b)(3a - 4b) =$
5. $(7a + x)(7a - x) =$
6. $(7ab + x)(7ab - x) =$
7. $(8a + b)(8a - b) =$
8. $(8a + 7bc)(8a - 7bc) =$
9. $(5a^3 + 6b^3)(5a^3 - 6b^3) =$
10. $(5x^2y + 3xy^2)(5x^2y - 3xy^2) =$

The utility of the preceding theorems will be the more apparent, the more complicated the expressions to which they are applied. Frequent examples of their application will be seen hereafter.

(63.) The same theorems will enable us to resolve many complicated expressions into their factors.

1. Resolve $a^2 + 2ab + b^2$ into its factors.

Ans. $(a + b)(a + b.)$

2. Resolve $a^2 + 4ab + 4b^2$ into its factors.
3. Resolve $a^2 - 6ab + 9b^2$ into its factors.
4. Resolve $9a^2 - 24ab + 16b^2$ into its factors.

5. Resolve $a^2 - b^2$ into its factors.
6. Resolve $a^4 - b^4$ into three factors.
7. Resolve $a^6 - b^6$ into its factors.
8. Resolve $a^8 - b^8$ into four factors.
9. Resolve $25a^4 - 60a^2b^3 + 36b^6$ into its factors.
10. Resolve $n^2 + 2n + 1$ into its factors.
11. Resolve $4m^2n^2 - 4mn + 1$ into its factors.
12. Resolve $49a^4b^4 - 168a^3b^3 + 144a^2b^2$ into its factors.
13. Resolve $n^3 + 2n^2 + n$ into its factors.

MULTIPLICATION BY DETACHED COEFFICIENTS.

(64.) In the multiplication of polynomials, the letters may often be omitted, and the operation be performed on the coefficients alone. The proper letters are afterwards annexed to the coefficients in the result. This will be best understood from a few examples.

Thus, take the first example of Art. 52, to multiply $a + b$ by $a + b$.

$$\begin{array}{lrrr}
\text{The coefficients of the multiplicand are} & 1 & +1 & \\
\text{`` \quad `` \quad multiplier} & 1 & +1 & \\ \cline{2-3}
 & 1 & +1 & \\
 & & 1 & +1 \\ \cline{2-4}
\text{Coefficients of the product} & 1 & +2 & +1 \\
\text{or supplying the letters we obtain} & a^2 & +2ab & +b^2
\end{array}$$

the same result as before obtained.

Ex. 2. Multiply $3a^2 + 4ax - 5x^2$ by $2a^2 - 6ax + 4x^2$.

$$\begin{array}{lrrrrr}
\text{Coefficients of multiplicand} & 3 & +4 & -5 & & \\
\text{`` \quad multiplier} & 2 & -6 & +4 & & \\ \cline{2-4}
 & 6 & +8 & -10 & & \\
 & & -18 & -24 & +30 & \\
 & & & +12 & +16 & -20 \\ \cline{2-6}
\text{Coefficients of the product} & 6 & -10 & -22 & +46 & -20
\end{array}$$

It may seem difficult in this case to supply the letters; but a little consideration will render it perfectly plain. Thus $3a^2 \times 2a^2$ is equal to $6a^4$; hence a^4 is the proper letter to be attached to the first coefficient. For the same reason x^4 is the proper letter to be attached to the last coefficient. Moreover, we see that both the proposed polynomials are homogeneous, and of the second degree. Hence the product must be homogeneous, and of the fourth degree. The powers of a must decrease successively by unity, beginning with the first term, while those of x increase by unity. Hence, the required product is

$$6a^4 - 10a^3x - 22a^2x^2 + 46ax^3 - 20x^4.$$

Ex. 3. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$.

Ex. 4. Multiply $x^3 - 3x^2 + 3x - 1$ by $x^2 - 2x + 1$.

Ex. 5. Multiply $2a^3 - 3ab^2 + 5b^3$ by $2a - 5b$.

If we should proceed with this example precisely in the same manner as with the preceding, we should commit an error by attempting to unite terms which. are dissimilar. The reason is that the multiplicand does not contain the usual complete series of powers of a. The term containing the second power of a is wanting. This does not render the method inapplicable, but it is necessary to keep dissimilar terms distinct from each other; and since while we are operating on the coefficients, we have not the advantage of the letters to indicate what are similar terms, we supply the place of the deficient term by a cypher. The operation will then proceed with entire regularity.

$$\begin{array}{lllll}
2 & +0 & -3 & +5 & \\
2 & -5 & & & \\
\hline
4 & +0 & -6 & +10 & \\
 & -10 & -0 & +15 & -25 \\
\hline
4 & -10 & -6 & +25 & -25
\end{array}$$

Hence the product is

$$4a^4 - 10a^3b - 6a^2b + 25ab^3 - 25b^4.$$

Ex. 6. Multiply $2a^3 - 3ab^2 + 5b^3$ by $2a^2 - 5b^2$.

Here there is a term in each polynomial to be supplied by a cypher.

The preceding examples are useful in leading the student to consider the properties of coefficients by themselves, and as preparatory to some investigations which are to follow. The beginner, however, in attempting to apply the method, will need to be particularly cautious not to unite dissimilar terms.

SECTION V.

DIVISION.

(65.) The object of division in Algebra is the same as in Arithmetic, viz. *The product of two factors being given, and one of the factors, to find the other factor.*

The dividend is the product of the divisor and quotient, the divisor is the given factor, and the quotient is the factor required to be found.

CASE I.

(66.) *When the divisor and dividend are both monomials.*

Suppose we have 63 to be divided by 7. We must find such a factor, as multiplied by 7 will give exactly 63. We perceive that 9 is such a number, and therefore 9 is the quotient obtained when we divide 63 by 7.

So also, if we have to divide ab by a, it is evident that the quotient will be b; for a multiplied by b gives the dividend ab. It is plain also that if we have to divide ab by b, the quotient will be a.

Again the dividend abc divided by a gives bc; for a multiplied by bc gives abc. So also $12mn$ divided by $3m$ gives $4n$; for $3m$ multiplied by $4n$ makes $12mn$.

Suppose we have a^5 to be divided by a^2. We must find a number which multiplied by a^2 will produce a^5. We perceive that a^3 is such a number; for according to Art. 50 we multiply a^3

by a^2, by adding the exponents 2 and 3, making 5. That is, the exponent 3 of the quotient is found by subtracting 2 the exponent of the divisor from 5 the exponent of the dividend. Hence the following

RULE OF EXPONENTS IN DIVISION.

In order to divide quantities expressed by different powers of the same letter, *subtract the exponent of the divisor from the exponent of the dividend.*

EXAMPLES.

Divide	a^8	a^7	b^6	c^6	h^7	x^9	y^{11}
By	a^3	a^2	b^3	c^4	h^4	x^3	y^8
Quotient	a^5						

Let it be required to divide $35a^5$ by $5a^2$. We must find a quantity which multiplied by $5a^2$ will produce $35a^5$. Such a quantity is $7a^3$; for according to Arts. 49 and 50, $7a^3 \times 5a^2$ is equal to $35a^5$. Therefore $35a^5$ divided by $5a^2$ gives for a quotient $7a^3$; that is, we have divided 35, the coefficient of the dividend, by 5 the coefficient of the divisor; and have subtracted the exponent of the divisor from the exponent of the dividend.

(67.) Hence for the division of monomials, we have the following

RULE.

1. *Divide the coefficient of the dividend by the coefficient of the divisor.*

2. *Subtract the exponents of the letters in the divisor from the exponents of the same letters in the dividend.*

EXAMPLES.

1. Divide $20x^3$ by $4x$ Ans. $5x^2$
2. Divide $25a^3xy^4$ by $5ay^2$
3. Divide $72ab^5x^2$ by $12b^3x$
4. Divide $77a^3b^5c^6$ by $11ab^3c^4$

5. Divide $272a^3b^4c^5x^6$ by $17a^2b^3cx^4$

6. Divide $250x^7y^8z^3$ by $5xyz^3$

7. Divide $48a^3b^3c^2d$ by $12ab^2\mathrm{c}$

8. Divide $150a^5b^8cd^3$ by $30a^3b^5d^2$

(68.) The rule given in Art. 66 conducts in some cases to *negative exponents.*

Thus, let it be required to divide a^3 by a^5. We are directed to subtract the exponent of the divisor from the exponent of the dividend. We thus obtain

$$a^{3-5} = a^{-2}$$

But a^3 divided by a^5 may be written $\frac{a^3}{a^5}$; and since the value of a fraction is not altered by dividing both numerator and denominator by the same quantity, this expression is equivalent to $\frac{1}{a^2}$.

Hence a^{-2} is the same as $\frac{1}{a^2}$

and these expressions may be used indifferently for each other.

So also, if a^2 is to be divided by a^5, this may be written

$$\frac{a^2}{a^5} = \frac{1}{a^3} = a^{-3}$$

In the same manner we find

$$a^{-m} = \frac{1}{a^m}$$

That is, *the reciprocal of a quantity is equal to the same quantity with the sign of its exponent changed.*

So also $$\frac{a}{b^2c} = \frac{ab^{-2}}{c} = ab^{-2}c^{-1}$$

And $$\frac{ad^{-4}}{b} = \frac{a}{bd^4}$$

(69.) Hence any factor may be transferred from the numerator to the denominator of a fraction, or from the denominator to the numerator, by *changing the sign of its index.*

Thus $$\frac{a}{b} = ab^{-1}$$

$$\frac{a^{-2}}{b^{-2}} = a^{-2}b^2$$

$$\frac{a^4b^{-m}}{c^2d^{-3}} = a^4b^{-m}c^{-2}d^3$$

That is, the denominator of a fraction may be entirely removed, and an *integral* form be given to any fractional expression.

This use of negative exponents must be understood simply as a convenient notation, and not as a method of actually destroying the denominator of a fraction. Still this new notation has many advantages, and is often employed, as will be seen hereafter.

When the division cannot be exactly performed, it may be expressed in the form of a fraction, and this fraction may be reduced to its lowest terms according to a method to be explained in Art. 83.

(70.) It frequently happens that the exponents of certain letters in the dividend are the *same* as in the divisor.

Let it be required to divide a^2 by a^2. The quotient is obviously 1, for every number is contained in itself once. But if we apply the rule of exponents, Art. 66, we shall have

$$a^{2-2} \text{ or } a^0.$$

Hence $$a^0 = 1.$$

Again let it be required to divide a^m by a^m. The quotient is obviously 1 as before; and applying the rule of exponents we obtain

$$a^{m-m} \text{ or } a^0.$$

That is, *every quantity affected with the exponent zero, is equal to unity.*

This notation has the advantage of preserving a trace of a letter which has disappeared in the operation of division. Thus let it be required to divide a^3b^2 by a^2b^2. The quotient will be ab^0. This expression is of the same value as a alone, and is commonly so written. If, however, it was important to indicate that the letter b originally entered into the expression, this might be done without at all affecting the value of the result by writing it

$$ab^0.$$

(71.) The proper *sign* to be prefixed to the quotient is readily deduced from the principles already established for multiplication. The product of the divisor and quotient must be equal to the dividend. Hence

$$\text{because}\left.\begin{array}{r} +a\times+b=+ab\\ -a\times+b=-ab\\ +a\times-b=-ab\\ -a\times-b=+ab\end{array}\right\}\text{therefore}\left\{\begin{array}{l} +ab\div+b=+a\\ -ab\div+b=-a\\ -ab\div-b=+a\\ +ab\div-b=-a\end{array}\right.$$

Hence we have the following

RULE FOR THE SIGNS.

When both the dividend and divisor have the same sign, the quotient will have the sign + ; when they have different signs, the quotient will have the sign —.

EXAMPLES.

1. Divide $-15ay^2$ by $3ay$.
2. Divide $-18ax^2y$ by $-9ax$.
3. Divide $150a^3bc$ by $-5ac$.
4. Divide $40a^3b^4c$ by $-abc$.

CASE II.

(72.) *When the divisor is a monomial, and the dividend a polynomial.*

We have seen, Art. 51, that when a single term is multiplied into a polynomial, the former enters into *every term* of the latter.

Thus $a(a+b)=a^2+ab$

Hence $(a^2+ab)\div a=a+b$.

Whence we deduce the following

RULE.

Divide each term of the dividend by the divisor, as in the former case.

EXAMPLES.

1. Divide $3x^3 + 6x^2 + 3ax - 15x$ by $3x$.
2. Divide $3abc + 12abx - 9a^2b$ by $3ab$.
3. Divide $40a^3b^3 + 60a^2b^2 - 17ab$ by $-ab$.
4. Divide $15a^2bc - 10acx^2 + 5ad^2$ by $-5a^2c$.
5. Divide $6a^2x^4y^6 - 12a^3x^3y^6 + 15a^4x^5y^3$ by $3a^2x^2y^2$.
6. Divide $x^{n}{}^{+1} - x^{n}{}^{+2} + x^{n}{}^{+3} - x_{n}{}^{+4}$ by x_n.
7. Divide $12a^4y^6 - 16a^5y^5 + 20a^6y^4 - 28a^7y^3$ by $-4a^4y^3$.

CASE III.

(73.) *When the divisor and dividend are both polynomials.*

Let it be required to divide $2ab + a^2 + b^2$ by $a + b$.

The object of this operation is to find a third polynomial which multiplied by the second will reproduce the first.

It follows from this definition, and from the rule of Art. 52, that the dividend is composed of all the partial products arising from the multiplication of each term of the divisor, by each term of the quotient; these products being added together and *reduced*. Hence if we can discover a term of the dividend which is derived without *reduction* from the multiplication of a term of the divisor by a term of the quotient, then dividing this term by the corresponding term of the divisor, we shall be sure to obtain a term of the quotient.

But from Art. 58, it appears that the term a^2, which contains the *highest exponent* of the letter a, is derived *without reduction* from the multiplication of the two terms of the divisor and quotient which are affected with the highest exponent of the same letter. Dividing then the term a^2 by the term a of the divisor, we obtain a which we are certain must be one term of the quotient sought. Multiplying each term of the divisor by a, and subtracting this product from the proposed dividend, the remainder may be regarded as the product of the divisor by the remaining terms of the quotient. We shall then obtain another term of the quotient, by dividing that term of the remainder affected with the highest exponent of a, by the term a of the divisor, and so on.

Thus we perceive that at each step, we are obliged to search for that term of the dividend which is affected with the highest exponent of one of the letters, and divide it by that term of the divisor which is affected with the highest exponent of the same letter. We may avoid the necessity of *searching* for this term, by arranging the terms of the divisor and dividend *in the order of the powers of one of the letters.*

The operation will then proceed as follows:

$$\begin{array}{r|l} \text{The arranged dividend} = a^2 + 2ab + b^2 & a + b = \text{the divisor} \\ \cline{2-2} a^2 + ab & \\ \cline{1-1} & a + b = \text{the quotient} \\ ab + b^2 = \text{first remainder} & \\ ab + b^2 & \\ \cline{1-1} 0 & \end{array}$$

It is generally most convenient in Algebra to place the divisor on the right of the dividend, and the quotient directly under the divisor.

(74.) From this investigation we deduce the following

RULE FOR THE DIVISION OF POLYNOMIALS.

1. *Arrange the dividend and divisor according to the powers of the same letter.*

2. *Divide the first term of the dividend by the first term of the divisor, the result will be the first term of the quotient.*

3. *Multiply the divisor by this term, and subtract the product from the dividend.*

4. *Divide the first term of the remainder by the first term of the divisor, the result will be the second term of the quotient.*

5. *Multiply the divisor by this term, and subtract the product from the last remainder. Continue the same operation, and if you obtain* 0 *for a remainder, the division is said to be exact.*

EXAMPLES.

1. Divide $2a^2b + b^3 + 2ab^2 + a^3$ by $a^2 + b^2 + ab$.

Ans. $a + b$.

2. Divide $x^3 - a^3 + 3a^2x - 3ax^2$ by $x - a$.
Ans. $x^2 - 2ax + a^2$.

3. Divide $a^6 + 2a^3z^3 + z^6$ by $a^2 - az + z^2$.
4. Divide $a^6 - 16a^3x^3 + 64x^6$ by $a^2 - 4ax + 4x^2$.
5. Divide $a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$ by $a^2 - 2ax + x^2$.
6. Divide $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.
7. Divide $a^2 + 4ax + 4x^2$ by $a + 2x$.
8. Divide $12x^4 - 192$ by $3x - 6$.
9. Divide $6x^6 - 6y^6$ by $2x^2 - 2y^2$.
10. Divide $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$ by $a^3 - 3a^2b + 3ab^2 - b^3$.
11. Divide $x^2 - y^2$ by $x - y$.
12. Divide $a^4 - b^4$ by $a - b$.

If the first term of the arranged dividend is not divisible by the first term of the arranged divisor, the complete division is *impossible.*

(75.) Hitherto we have supposed the terms of the quotient to be obtained by dividing that term of the dividend affected with the *highest* exponent of a certain letter. But from the second remark of Art. 58, it appears that the term of the dividend affected with the *lowest* exponent of any letter, is derived without reduction from the multiplication of a term of the divisor by a term of the quotient. Hence we may obtain a term of the quotient by dividing the term of the dividend affected with the lowest exponent of any letter, by the term of the divisor containing the lowest power of the same letter; and nothing prevents our operating upon the highest and lowest exponents of a certain letter alternately, in the same example.

(76.) From the examples of Art. 74, we perceive that $x^2 - y^2$ is divisible by $x - y$; and $a^4 - b^4$ is divisible by $a - b$. We shall find the same to hold true, whatever may be the value of the exponents of the two letters. Their difference will always be divisible by $a - b$.

Let us designate any exponent whatever, by the letter n, and let us divide $a^n - b^n$ by $a - b$.

$$\begin{array}{rr|l} & a^n - b^n & a - b \\ & a^n - ba^{n-1} & \overline{\phantom{a^{n-1}}} \\ \cline{2-2} & & a^{n-1} \\ \text{1st remainder} = & ba^{n-1} - b^n & \end{array}$$

Dividing a^n by a, we have by the rule of exponents a^{n-1} for the quotient. Multiplying $a - b$ by this quantity, and subtracting the product from the dividend, we have for the first remainder $ba^{n-1} - b^n$, a result which may be put under the form

$$b\,(a^{n-1} - b^{n-1}).$$

Now if after a division has been partially performed, the remainder is divisible by the divisor, it is obvious that the dividend is completely divisible by the divisor. Hence it follows that if $a^{n-1} - b^{n-1}$ is divisible by $a - b$, $a^n - b^n$ will also be divisible by $a - b$. That is to say, if the difference of the same powers of two quantities is divisible by their difference, the difference of the powers of the *next higher degree* is also divisible by that difference.

But we have seen that $a^4 - b^4$ is divisible by $a - b$. Therefore $a^5 - b^5$ is divisible by $a - b$; therefore also $a^6 - b^6$ is divisible by $a - b$, and so on. Hence,

The difference of two integral positive powers of the same degree, is divisible by the difference of their roots.

The quotients obtained by dividing the difference of the powers of two quantities by the difference of those quantities, follow a simple law. Thus

$$(a^2 - b^2) \div (a - b) = a + b$$
$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2$$
$$(a^4 - b^4) \div (a - b) = a^3 + a^2b + ab^2 + b^3$$
$$(a^5 - b^5) \div (a - b) = a^4 + a^3b + a^2b^2 + ab^3 + b^4.$$

&c. &c. &c.

$$(a^n - b^n) \div a - b) = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdot\cdot + a^2b^{n-3} + ab^{n-2} + b^{n-1}.$$

The exponent of a decreases by unity, while that of b increases by unity.

(77.) It may also be proved that *the difference of two powers of* *the same degree is divisible by the sum of their roots, when the exponent of the power is an even number.*

Thus,

$(a^2-b^2)\div(a+b)=a-b$
$(a^4-b^4)\div(a+b)=a^3-a^2b+ab^2-b^3$
$(a^6-b^6)\div(a+b)=a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5$
&c. &c. &c.

Also, *the sum of two powers of the same degree, is divisible by the sum of their roots, when the exponent of the power is an odd number.*

Thus,

$(a^3+b^3)\div(a+b)=a^2-ab+b^2$
$(a^5+b^5)\div(a+b)=a^4-a^3b+a^2b^2-ab^3+b^4$
$(a^7+b^7)\div(a+b)=a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6.$
&c. &c. &c.

(78.) The preceding principles will enable us to resolve various algebraic expressions into their factors.

1. Resolve a^3-b^3 into its factors.
Ans. $(a^2+ab+b^2)(a-b)$.
2. Resolve a^3+b^3 into its factors.
3. Resolve a^6-b^6 into four factors.
4. Resolve a^3-8b^3 into its factors.
5. Resolve $8a^3-b^3$ into its factors.
6. Resolve $8a^3-8b^3$ into its factors.
7. Resolve a^3+27b^3 into its factors.
8. Resolve $8a^3+27b^3$ into its factors.

(79.) One polynomial cannot be divided by another polynomial containing *a letter which is not found* in the dividend; for it is impossible that a third quantity multiplied by a second which contains a certain letter, should give a product *not containing* that letter.

A monomial is never divisible by a polynomial; because every polynomial multiplied by another quantity, gives a product containing at least *two terms* not susceptible of reduction.

Yet a binomial may be divided by a polynomial, containing *any number* of terms.

Thus, a^4-b^4 is divisible by $a^3+a^2b+ab^2+b^3$, and gives for a quotient $a-b$.

So also, a binomial may be divided by a polynomial of a hundred terms, a thousand terms, or indeed any finite number.

DIVISION BY DETACHED COEFFICIENTS.

(80.) We have shown in Art. 64 how multiplication may sometimes be conveniently performed by operating upon the coefficients alone. The same principle is applicable to division. Thus, take the example of Art. 73, to divide $a^2+2ab+b^2$ by $a+b$; we may proceed as follows:

$$
\begin{array}{r|l}
1+2+1 & 1+1 \\ \cline{2-2}
1+1 & 1+1 \\ \cline{1-1}
1+1 & \\
1+1 &
\end{array}
$$

The coefficients of the quotient are $1+1$. Moreover, $a^2 \div a = a$; and therefore a is the first term of the quotient, and b the second.

Ex. 2. Divide $x^4-3ax^3-8a^2x^2+18a^3x-8a^4$ by $x^2+2ax-2a^2$.

$$
\begin{array}{r|l}
1-3-8+18-8 & 1+2-2 \\ \cline{2-2}
1+2-2 & 1-5+4 \\ \cline{1-1}
-5-6+18-8 & \\
-5-10+10 & \\ \cline{1-1}
4+8-8 & \\
4+8-8 &
\end{array}
$$

The coefficients of the quotient are $1-5+4$, and it remains to supply the letters. But $x^4 \div x^2 = x^2$; and $a^4 \div a^2 = a^2$. Hence x^2, ax, and a^2 are the literal parts of the terms, and therefore the quotient is

$$x^2-5ax+4a^2.$$

Ex. 3. Divide $6a^4-96$ by $3a-6$.

Here as we have the fourth power of a without the lower powers, we must supply the coefficients of the absent terms, as in multiplication, with zero.

$$\begin{array}{llll|l}
6+0+0+0-96 & & & & 3-6 \\
\cline{5-5}
6-12 & & & & 2+4+8+16 \\
\cline{1-1}
\quad 12 & & & & \\
\quad 12-24 & & & & \\
\cline{1-1}
\qquad 24 & & & & \\
\qquad 24-48 & & & & \\
\cline{1-1}
\qquad\quad 48-96 & & & & \\
\qquad\quad 48-96 & & & &
\end{array}$$

But $a^4 \div a = a^3$; hence the quotient is

$$2a^3 + 4a^2 + 8a + 16.$$

Ex. 4. Divide $8a^5 - 4a^4x - 2a^3x^2 + a^2x^3$ by $4a^2 - x^2$.

Ex. 5. Divide $3y^3 + 3xy^2 - 4x^2y - 4x^3$ by $x + y$.

Ex. 6. Divide $a^6 + 4a^5 - 8a^4 - 25a^3 + 35a^2 + 21a - 28$ by $a^2 + 5a + 4$.

SECTION VI.

FRACTIONS.

(81.) When a quotient is expressed as described in Art. 16, by placing the divisor under the dividend with a line between them, it is called a *fraction;* the dividend is called the numerator and the divisor the denominator of the fraction. Algebraic fractions do not differ essentially from arithmetical fractions, and the same principles are applicable to both.

The following principles are the basis of most of the operations upon fractions:

1. *In order to multiply a fraction by any number, we must multiply the numerator, or divide the denominator of the fraction by that number.*

Thus the value of the fraction $\frac{ab}{a}$ is b. If we multiply the numerator by a, we obtain $\frac{a^2b}{a}$ or ab; and if we divide the denominator of the same fraction by a, we obtain also ab; that is, the original value of the fraction b, has been multiplied by a.

2. *In order to divide a fraction by any number, we must divide the numerator, or multiply the denominator by that number.*

Thus the value of the fraction $\frac{a^2b}{a}$ is ab. If we divide the numerator by a, we obtain $\frac{ab}{a}$ or b; and if we multiply the denominator of the same fraction by a we obtain $\frac{a^2b}{a^2}$ or b; that is, the original value of the fraction ab has been divided by a.

3. *The value of a fraction is not changed, if we multiply or divide both numerator and denominator by the same number.*

Thus $$\frac{ab}{a}=\frac{abx}{ax}=\frac{abxy}{axy}=b.$$

Every quantity which is not expressed under a fractional form, is called an *entire* quantity.

An algebraic expression composed partly of an entire quantity, and partly of a fraction, is called a *mixed* quantity.

(82.) The proper *sign* to be prefixed to a fraction, may be determined by the rules already established for division. The sign prefixed to the numerator of a fraction, affects merely the *dividend*; the sign prefixed to the denominator, affects merely the *divisor*; but the sign prefixed to the dividing line of a fraction, affects the *quotient*.

Thus $\frac{ab}{a}=+b$, for + divided by + gives +.

$\frac{-ab}{a}=-b$, for — divided by + gives —

$\frac{ab}{-a}=-b$, for + divided by — gives —

$\frac{-ab}{-a}=+b$, for — divided by — gives +

So also $-\frac{ab}{a}=-b$, for this shows that the former quotient b is to be subtracted, which is done by changing its sign.

$-\frac{-ab}{a}=+b$, because the former quotient $-b$ is to be subtracted, whence it becomes $+b$.

$-\frac{ab}{-a}=+b$, for the same reason;

and $-\frac{-ab}{-a}=-b$, also for the same reason.

Hence $\frac{ab}{a}=\frac{-ab}{-a}=-\frac{-ab}{a}=-\frac{ab}{-a}=+b$

So also, $\frac{-ab}{a}=\frac{ab}{-a}=-\frac{ab}{a}=-\frac{-ab}{-a}=-b$

Hence it appears that of the three signs belonging to the numerator, denominator, and dividing line of a fraction, *any two may be changed from + to — or from — to +, without affecting the value of the fraction.*

In the examples of fractions here employed for illustration, both numerator and denominator have consisted of monomials. The same principles are applicable to polynomials; but it must be remarked that by the sign of the numerator, we understand the *entire* numerator as distinguished from the sign of any one of its terms taken *singly*.

When *no* sign is prefixed either to the terms of a fraction, or to its dividing line, + is always to be understood.

REDUCTION OF FRACTIONS.

CASE I.

(83.) *To reduce a fraction to lower terms.*

Divide both numerator and denominator by any quantity which will divide them both without a remainder.

According to Remark 3 of Art. 81, this will not change the *value* of the fraction.

Thus $\frac{ax}{bx}=\frac{a}{b}$

$\frac{a^2bc}{5a^2b^2}=\frac{c}{5b}$ (dividing both numerator and denominator by a^2b.)

$$\frac{ax^2}{ax+x^2}=\frac{ax}{a+x}.$$

If the numerator and denominator are both divided by their *greatest* common divisor, it is evident the fraction will be reduced to its *lowest* terms. The method of finding the greatest common divisor is considered in Section XV.

EXAMPLES.

1. Reduce $\frac{cx + x^2}{a^2c + a^2x}$ to its lowest terms.

Ans. $\frac{x}{a^2}$

2. Reduce $\frac{x^2 - a^2}{x^4 - a^4}$ to its lowest terms.

3. Reduce $\frac{2x^3 - 16x - 6}{3x^3 - 24x - 9}$ to its lowest terms.

4. Reduce $\frac{3x^2y + 3xy^2}{3x^2 + 6xy + 3y^2}$ to its lowest terms.

5. Reduce $\frac{a^2 - b^2}{a^2 - 2ab + b^2}$ to its lowest terms.

6. Reduce $\frac{a^3 - x^3}{a^2 - 2ax + x^2}$ to its lowest terms.

CASE II.

(84.) *To reduce a fraction to an entire or mixed quantity.*

RULE.

Divide the numerator by the denominator for the entire part, and place the remainder, if any, over the denominator for the fractional part.

Thus $\frac{27}{5} = 27 \div 5 = 5\frac{2}{5}$

Also $\frac{ax + a^2}{x} = (ax + a^2) \div x = a + \frac{a^2}{x}$.

EXAMPLES.

1. Reduce $\frac{ax - x^2}{x}$ to an entire quantity.

2. Reduce $\frac{ab-2a^2}{b}$ to a mixed quantity.

3. Reduce $\frac{a^2+x^2}{a-x}$ to a mixed quantity.

4. Reduce $\frac{x^3-y^3}{x-y}$ to an entire quantity.

5. Reduce $\frac{10x^2-5x+3}{5x}$ to a mixed quantity.

6. Reduce $\frac{8b^3-16b+7a^2b^2}{8b}$ to a mixed quantity.

CASE III.

(85.) *To reduce a mixed quantity to the form of a fraction.*

RULE.

Multiply the entire part by the denominator of the fraction; to the product add the numerator with its proper sign; and place the result over the denominator.

Thus $3\frac{2}{5}=\frac{3\times5+2}{5}=\frac{15+2}{5}=\frac{17}{5}$.

This result may be proved by the preceding case. For $\frac{17}{5}=17\div5=3\frac{2}{5}$.

Also $a-\frac{b}{c}=\frac{a\times c-b}{c}=\frac{ac-b}{c}$.

EXAMPLES.

1. Reduce $x-\frac{a^2-x^2}{x}$ to the form of a fraction.

2. Reduce $x - \frac{ax + x^2}{2a}$ to the form of a fraction.

3. Reduce $5 + \frac{2x - 7}{3x}$ to the form of a fraction.

4. Reduce $1 - \frac{x - a - 1}{a}$ to the form of a fraction.

5. Reduce $1 + 2x - \frac{x - 3}{5x}$ to the form of a fraction.

6. Reduce $7 + \frac{3b^2 - 8c^2}{a^2 - b^2}$ to the form of a fraction.

CASE IV.

(86.) *To reduce fractions to a common denominator.*

RULE.

Multiply each numerator into all the denominators, except its own, for a new numerator, and all the denominators together for a common denominator.

EXAMPLES.

1. Reduce $\frac{a}{b}$ and $\frac{c}{d}$ to a common denominator.

Ans. $\frac{ad}{bd}$; $\frac{bc}{bd}$.

Here it will be seen that the numerator and denominator of the first fraction are both multiplied by d, and in the second fraction, they are both multiplied by b. The *value* of the fractions therefore, is not changed by this operation.

2. Reduce $\frac{a}{b}$ and $\frac{a + b}{c}$ to equivalent fractions, having a common denominator.

3. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$ and d, to fractions having a common denominator.

4. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a + \frac{4x}{5}$ to fractions having a common denominator.

5. Reduce $\frac{a}{2}$, $\frac{3x}{7}$ and $\frac{a+x}{a-x}$ to fractions having a common denominator.

6. Reduce $\frac{x}{3}$, $\frac{x+1}{5}$ and $\frac{1-x}{1+x}$ to fractions having a common denominator.

7. Reduce $\frac{2a}{3x^2}$ and $\frac{a+2x}{4x}$ to fractions having a common denominator.

Following the Rule we obtain

$$\frac{8ax}{12x^3} \text{ and } \frac{3ax^2+6x^3}{12x^3}$$

which fractions have a common denominator and are equivalent to those originally proposed. Nevertheless it may be observed that these fractions are not reduced to their *least* common denominator, for every term is divisible by x. The least common denominator, is the least common multiple of the denominators of the proposed fractions.

A common multiple of two or more numbers is any number which they will divide without a remainder; and the *least* common multiple is the least number which they will so divide. Thus $12x^2$ is the least common multiplier of $3x^2$ and $4x$; and the above fractions reduced to their least common denominator are

$$\frac{8a}{12x^2} \text{ and } \frac{3ax+6x^2}{12x^2}.$$

The least common multiple of two numbers is their product divided by their greatest common divisor.

8. Reduce $\frac{3}{14}$ and $\frac{5}{21}$ to equivalent fractions having the least common denominator.

The product of the denominators is 294, which divided by 7 (their greatest common divisor) gives 42, the least common denominator, and the required fractions are

$$\frac{9}{42} \text{ and } \frac{10}{42}$$

9. Reduce the fractions $\frac{7}{10}$ and $\frac{11}{15}$ to others which have the least common denominator.

10. Reduce $\frac{a+b}{a-b}$ and $\frac{c+d}{a^2-b^2}$, to equivalent fractions having the least common denominator.

11. Reduce $\frac{2a}{3bc}$ and $\frac{bd}{6b^2c^2}$ to equivalent fractions having the least common denominator.

CASE V.

(87.) *To add fractional quantities together.*

RULE.

Reduce the fractions to a common denominator; add the numerators together, and place their sum over the common denominator.

EXAMPLES.

1. What is the sum of $\frac{x}{2}$ and $\frac{x}{3}$?

Reducing to a common denominator, the fractions become

$$\frac{3x}{6} \text{ and } \frac{2x}{6}.$$

Adding the numerators, we obtain $\frac{5x}{6}$.

It is plain that three sixths of x and two sixths of x make five sixths of x.

2. Required the sum of $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$.

Ans. $\frac{adf+bcf+bde}{bdf}$.

3. Required the sum of $\frac{x}{2}$, $\frac{x}{3}$ and $\frac{x}{4}$

4. Required the sum of $5x$, $\frac{2a}{3x^2}$ and $\frac{a+2x}{4x}$.

5. Required the sum of $2a$, $3a+\frac{2x}{5}$ and $a+\frac{8x}{9}$.

6. Required the sum of $a+x$, $\frac{a}{a-x}$ and $\frac{a-x}{a}$

7. Required the sum of $\frac{a+b}{2}$ and $\frac{a-b}{2}$.

CASE VI

(88.) *To subtract one fractional quantity from another.*

RULE.

Reduce the fractions to a common denominator, subtract one numerator from the other, and place their difference over the common denominator.

EXAMPLES.

1. From $\frac{2x}{3}$ subtract $\frac{3x}{5}$.

Reducing to a common denominator, the fractions become

$$\frac{10x}{15} \text{ and } \frac{9x}{15}.$$

Hence $$\frac{10x}{15} - \frac{9x}{15} = \frac{x}{15},$$

and it is plain that *ten* fifteenths of x diminished by *nine* fifteenths of x, equals *one* fifteenth of x.

2. From $\frac{12x}{7}$ subtract $\frac{3x}{5}$.

3. From $\frac{5y}{3}$ subtract $\frac{3y}{8}$.

4. From $\frac{ax}{b-c}$ subtract $\frac{ax}{b+c}$.

5. From $2x + \frac{2+7x}{8}$ subtract $x - \frac{5x-6}{21}$.

6. From $3x + \frac{x}{2b}$ subtract $x - \frac{x-a}{c}$.

7. From $\frac{a+b}{2}$ subtract $\frac{a-b}{2}$.

8. From $\frac{13a-5b}{4}$ subtract $\frac{7a-2b}{6}$.

CASE VII.

(89.) *To multiply fractional quantities together.*

RULE.

Multiply all the numerators together for a new numerator, and all the denominators together for a new denominator.

Let it be required to multiply $\frac{a}{b}$ by $\frac{c}{d}$.

First, let us multiply $\frac{a}{b}$ by c. According to Remark first of Art. 81, the product must be $\frac{ac}{b}$.

But the proposed multiplier was $\frac{c}{d}$, that is, we have used a multiplier d times too great. We must therefore, divide the result $\frac{ac}{b}$ by d; and according to Remark second of Art. 81, we obtain

$$\frac{ac}{bd},$$

which result conforms to the Rule above given.

Ex. 1. Multiply $\frac{x}{6}$ by $\frac{2x}{9}$.

Ans. $\frac{x^2}{27}$.

2. What is the continued product of $\frac{x}{2}$, $\frac{4x}{5}$, and $\frac{10x}{21}$?

3. Multiply $\frac{x}{a}$ by $\frac{x+a}{a+c}$.

4. What is the continued product of $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{3ac}{2b}$?

5. Multiply $b + \frac{bx}{a}$ by $\frac{a}{x}$.

6. Multiply $\frac{x^2 - b^2}{bc}$ by $\frac{x^2 + b^2}{b + c}$.

7. What is the continued product of $x, \frac{x + 1}{a}$ and $\frac{x - 1}{a + b}$?

8. Multiply $\frac{a^2 + b^2}{a^2 - b^2}$ by $\frac{a - b}{a + b}$.

(90.) Ex. 1. Multiply $\frac{1}{a^3}$ by $\frac{1}{a^2}$.

According to the preceding Article, the result must be $\frac{1}{a^5}$.

But according to Art. 68, $\frac{1}{a^3}$ may be written a^{-3}; $\frac{1}{a^2}$ may be written a^{-2}; and $\frac{1}{a^5}$ may be written a^{-5}.

Therefore $a^{-3} \times a^{-2} = a^{-5}$.

That is, *the Rule of Art.* 50 *is general, and applies to negative as well as positive exponents.*

Ex. 2. Multiply $-b^{-2}$ by b^{-3}.

Ans. $-b^{-5}$.

3. Multiply a^{-2} by a^3.
4. Multiply b^{-3} by b^3.
5. Multiply a^{-n} by a^{-m}.
6. Multiply b^{-n} by b^m.

CASE VIII.

(91.) *To divide one fractional quantity by another.*

RULE.

Invert the divisor and proceed as in multiplication.

If the two fractions have the same denominator, then the quotient of the fractions will be the same as the quotient of their numerators.

Thus it is plain that $\frac{3}{12}$ is contained in $\frac{9}{12}$ as often as 3 is contained in 9.

But when the two fractions have not the same denominator, we must reduce them to this form by Case IV.

Let it be required to divide $\frac{a}{b}$ by $\frac{c}{d}$.

Reducing to a common denominator, we have $\frac{ad}{bd}$ to be divided by $\frac{bc}{bd}$.

It is now plain that the quotient must be represented by the division of *ad* by *bc*, which gives

$$\frac{ad}{bc}$$

the same result as obtained by the above Rule.

Thus $$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

EXAMPLES.

1. Divide $\frac{x}{3}$ by $\frac{2x}{9}$.

Ans. 1½.

2. Divide $\frac{2a}{b}$ by $\frac{4c}{d}$.

3. Divide $\frac{2x^2}{a^3+x^3}$ by $\frac{x}{x+a}$.

4. Divide $\frac{x+1}{6}$ by $\frac{2x}{3}$.

5. Divide $\frac{x-b}{8cd}$ by $\frac{3cx}{4d}$.

6. Divide $\frac{2ax+x^2}{c^3-x^3}$ by $\frac{x}{c-x}$.

7. Divide $\frac{a}{a+b}+\frac{b}{a-b}$ by $\frac{a}{a-b}-\frac{b}{a+b}$.

(92.) Ex. 1. Divide $\frac{1}{a^5}$ by $\frac{1}{a^3}$.

According to the Rule of the preceding Article, we have

$$\frac{1}{a^5}\times\frac{a^3}{1}=\frac{a^3}{a^5}=\frac{1}{a^2}.$$

But $\frac{1}{a^5}$ may be written a^{-5}; $\frac{1}{a^3}$ may be written a^{-3}; and $\frac{1}{a^2}$ is equal to a^{-2}.

Hence $$a^{-5}\div a^{-3}=a^{-2}.$$

That is, *the Rule of Art.* 66 *is general, and applies to negative as well as positive exponents.*

Ex. 2. Divide $-b^{-5}$ by $-b^{-2}$.

Ans. b^{-3}.

3. Divide a^2 by a^{-1}.
4. Divide 1 by a^{-4}.
5. Divide $6a^n$ by $-2a^{-3}$.
6. Divide b^{m-n} by b^m.
7. Divide $12x^{-2}y^{-4}$ by $-4xy^2$.

(93.) According to the definition, Art. 33, the reciprocal of a quantity is the quotient arising from dividing a unit by that quantity.

Hence the reciprocal of $\frac{a}{b}$

$$\text{is } 1 \div \frac{a}{b} = 1 \times \frac{b}{a} = \frac{b}{a}.$$

That is, *the reciprocal of a fraction is the fraction inverted.*

Thus the reciprocal of $\frac{a}{b+x}$ is $\frac{b+x}{a}$

The reciprocal of $\frac{1}{b+c}$ is $b+c$.

Hence, *to divide by any quantity, is the same as to multiply by its reciprocal; and to multiply by a quantity, is the same as to divide by its reciprocal.*

(94.) The numerator or denominator of a fraction may be itself a fraction;

As $$\frac{\frac{a}{b}}{c} \text{ or } \frac{\frac{a}{b}}{\frac{c}{d}}.$$

Such expressions are easily reduced by applying the preceding principles.

Thus $$\frac{\left\{\frac{a}{b}\right\}}{c} \text{ means } \frac{a}{b} \div c$$

which, according to Remark second, Art. 81, equals $\frac{a}{bc}$.

Again, $$\frac{a}{\left\{\frac{b}{c}\right\}} \text{ means } a \div \frac{b}{c}$$

which according to Art. 91, equals $\frac{ac}{b}$.

Also $\dfrac{\left\{\dfrac{a}{b}\right\}}{\left\{\dfrac{c}{d}\right\}}$ means the same as $\dfrac{a}{b} \div \dfrac{c}{d}$.

which, according to Art. 91, equals $\dfrac{ad}{bc}$.

SECTION VII.

SIMPLE EQUATIONS.

(95.) *An equation is a proposition which declares the equality of two quantities expressed algebraically.*

Thus $x - 4 = b - x$, is a proposition expressing the equality of the quantities $x - 4$ and $b - x$.

The quantity on the left hand side of the sign of equality is called the *first member* of the equation; the quantity on the right, the *second member*.

Equations are usually composed of certain quantities which are *known*, and others which are *unknown*. The known quantities are represented either by numbers, or by the first letters of the alphabet, a, b, c, &c.; the unknown quantities by the last letters, x, y, z, &c.

An *identical equation*, is one in which the two members are identical, or may be reduced to identity by performing the operations which are indicated in them.

Thus
$$2x - 5 = 2x - 5$$
$$3x + 4x = 7x$$
$$(x + y)(x - y) = x^2 - y^2.$$

A *root* of an equation, is the value of the unknown quantity in the equation.

(96.) Equations are divided into *degrees*, according to the highest power of the unknown quantity which they contain.

Those which contain only the *first* power of the unknown quantity, are called *simple equations*, or equations of *the first degree.*

$$\text{As } ax + b = cx + d.$$

Those in which the highest power of the unknown quantity is a *square*, are called *quadratic equations*, or equations of *the second degree.*

As $$4x^2 - 2x = 5 - x^2.$$

Those in which the highest power is a *cube* are called *cubic equations*, or equations of *the third degree.*

As $$x^3 + px^2 = 2q.$$

So also, we have *biquadratic equations*, or equations of *the fourth degree;* equations of the *fifth*, *sixth*, — — — — — *n*th degree.

Thus $x^n + px^{n-1} = r$, is an equation of the *n*th degree.

In general, *the degree of an equation is determined by the highest of the exponents with which the unknown quantity is affected.*

(97.) *Numerical equations* are those which contain only particular numbers, with the exception of the unknown quantity, which is always denoted by a letter.

Thus, $x^3 + 4x^2 = 3x + 12$ is a numerical equation.

Literal equations are those in which the known quantities are represented by letters, or by letters and numbers.

$$\text{Thus, } \left.\begin{array}{r} x^3 + px^2 + qx = r \\ x^4 - 3px^3 + 5qx^2 = 5 \end{array}\right\} \text{ are literal equations.}$$

To *solve* an equation, is to *find the value of the unknown quantity;* or to find a number which, substituted for the unknown quantity in the equation, renders the first member identical with the second.

The difficulty of solving equations, depends upon their *degree*, and *the number of unknown quantities.* We will begin with the most simple case.

(98.) SIMPLE EQUATIONS CONTAINING BUT ONE UNKNOWN QUANTITY.

The various operations which we perform upon equations in order to deduce the value of the unknown quantities, are founded upon the following principles:

1. If to two equal quantities, the same quantity be *added*, the sums will be equal.

2. If from two equal quantities, the same quantity be *subtracted*, the remainders will be equal.

3. If two equal quantities be *multiplied* by the same quantity, the products will be equal.

4. If two equal quantities be *divided* by the same quantity, the quotients will be equal.

(99.) The unknown quantity may be combined with the known quantities in the given equation, by the operations of *addition*, *subtraction*, *multiplication*, or *division*.

We shall consider these different cases in succession.

I. The unknown quantity may be combined with known quantities, by *addition*.

Let it be required to solve the equation

$$x + 6 = 24.$$

If from the two equal quantities, $x + 6$ and 24, we subtract the same quantity 6, the remainders will be equal, according to the last Article, and we shall have

$$x + 6 - 6 = 24 - 6$$

$$\text{or } x = 24 - 6$$

$$= 18 \text{ the value of } x \text{ required.}$$

So also in the equation

$$x + a = b,$$

subtracting a from each of the equal quantities, $x + a$, and b, the result is

$$x = b - a, \text{ the value of } x \text{ required.}$$

(100.) II. The unknown quantity may be combined with known quantities by *subtraction*.

Let the equation be

$$x - 6 = 24.$$

If to the two equal quantities $x - 6$ and 24, the same quantity 6 be added, the sums will be equal, according to Art. 98, and we have

$$x - 6 + 6 = 24 + 6$$

or $x = 30$, the value of x required.

So also in the equation

$$x - a = b,$$

adding a to each of these equal quantities, the result is

$x = b + a$, the value of x required.

From the preceding examples, it follows that

We may transpose any term of an equation from one member to the other by changing its sign.

We may change the sign of every term of an equation, without destroying the equality.

This is in fact the same thing as transposing *every term* in each member of the equation.

If the same quantity appear in each member of the equation affected with the same sign, it may be suppressed.

(101.) III. The unknown quantity may be combined with known quantities by *multiplication.*

Let the equation be

$$6x = 24.$$

If we divide each of the equal quantities, $6x$ and 24, by the same quantity 6, the quotients will be equal, and we shall have

$$x = \frac{24}{6}$$

$= 4$, the value of x required.

So also in the equation

$$ax = b,$$

dividing each of these equals by a, the result is

$x = \frac{b}{a}$, the value of x required.

From this it follows that

When the unknown quantity is multiplied by a known quantity, the equation is solved by dividing both members by this known quantity.

(102.) IV. The unknown quantity may be combined with known quantities by *division.*

Let the equation be

$$\frac{x}{6} = 24.$$

If we multiply each of the equal quantities, $\frac{x}{6}$ and 24, by the same quantity 6, the products will be equal, and we shall have

$x = 144$, the value of x required.

So also in the equation

$$\frac{x}{a} = b,$$

multiplying each of these equals by a, the result is

$x = ab$, the value of x required.

From this it follows that,

When the unknown quantity is divided by a known quantity, the equation is solved by multiplying both members by this known quantity.

(103.) V. Several terms of an equation may be *fractional.*

Let the equation be

$$\frac{x}{2} = \frac{2}{3} + \frac{4}{5}.$$

Multiplying each of these equals by 2, the result is

$$x = \frac{4}{3} + \frac{8}{5}.$$

Multiplying each of these last equals by 3, we obtain

$$3x = 4 + \frac{24}{5};$$

and multiplying again by 5, we obtain

$$15x = 20 + 24,$$

an equation free from fractions.

We might have obtained the same result, by multiplying the original equation at once by the product of all the denominators.

Thus multiplying by 2 × 3 × 5, we have

$$\frac{30x}{2}=\frac{60}{3}+\frac{120}{5},$$

or reducing, we have

$$15x=20+24, \text{ as before.}$$

So also in the equation

$$\frac{x}{a}=\frac{b}{c}+\frac{d}{e},$$

multiplying successively by all the denominators, or by $a\,c\,e$ at once, we obtain

$$\frac{acex}{a}=\frac{abce}{c}+\frac{acde}{e}.$$

Cancelling from each term the letter which is common to its numerator and denominator, we have

$$cex=abe+acd;$$

an equation clear of fractions.

Hence it appears that

An equation may be cleared of fractions, by multiplying each member into all the denominators.

(104.) From the preceding remarks we deduce the following

RULE FOR THE SOLUTION OF A SIMPLE EQUATION CONTAINING ONE UNKNOWN QUANTITY.

1. *Clear the equation of fractions, and perform in both members all the algebraic operations indicated.*
2. *Transpose all the terms containing the unknown quantity to one side, and all the remaining terms to the other side of the equation, and reduce each member to its most simple form.*
3. *Divide each member by the coefficient of the unknown quantity.*

EXAMPLES.

1. Given $5x+8=4x+10$, to find the value of x.

Transposing $4x$ to the first member of the equation, and 8 to the second member, taking care to change their signs, (Art. 100) we have

$$5x-4x=10-8.$$

Uniting similar terms, $x = 2$.

In order to verify this result, put 2 in the place of x wherever it occurs in the original equation, and we shall obtain

$$5 \times 2 + 8 = 4 \times 2 + 10.$$

That is $\quad 10 + 8 = 8 + 10$

or $\quad 18 = 18$

an identical equation, which proves that we have found the correct value of x.

2. Given $x - 7 = \frac{x}{5} + \frac{x}{3}$, to find the value of x.

Multiplying every term of the equation by 5 and also by 3, in order to clear it of fractions, (Art. 103) we obtain

$$15x - 105 = 3x + 5x.$$

Hence, by transposition

$$15x - 3x - 5x = 105$$

or $\quad 7x = 105$

and therefore $\quad x = \frac{105}{7} = 15.$

To verify this result, put 15 in the place of x in the original equation, and we have

$$15 - 7 = \frac{15}{5} + \frac{15}{3}.$$

That is, $\quad 15 - 7 = 3 + 5$

or $\quad 8 = 8$

an identical equation.

3. Given $3ax - 4ab = 2ax - 6ac$, to find the value of x in terms of b and c.

Dividing every term by a, we have

$$3x - 4b = 2x - 6c.$$

By transposition

$$3x - 2x = 4b - 6c$$

or $\quad x = 4b - 6c.$

This result may be verified in the same manner as the preceding.

4. Given $3x^2 - 10x = 8x + x^2$ to find the value of x.

Ans. $x = 9$.

5. Given $\frac{x}{2} + \frac{x}{3} = \frac{x}{4} + 7$ to find the value of x.

Ans. $x = 12$.

6. Given $\frac{x-5}{4} + 6x = \frac{284-x}{5}$ to find x.

7. Given $x + \frac{11-x}{3} = \frac{19-x}{2}$ to find x.

8. Given $3x + \frac{2x+6}{5} = 5 + \frac{11x-37}{2}$ to find x.

9. Given $\frac{6x-4}{3} - 2 = \frac{18-4x}{3} + x$ to find the value of x.

10. Given $21 + \frac{3x-11}{16} = \frac{5x-5}{8} + \frac{97-7x}{2}$ to find x.

(105.) An equation may always be cleared of fractions by multiplying each member into *all* the denominators according to Art. 103. But sometimes the same object may be attained by a less amount of multiplication.

Thus in the preceding example, the equation may be cleared of fractions by multiplying each term by 16, instead of $16 \times 8 \times 2$, and it is important to avoid all useless multiplication. In general, it is sufficient to multiply by the *least common multiple* of all the denominators. See Art. 86.

11. Given $x + \frac{3x-5}{2} = 12 - \frac{2x-4}{3}$ to find the value of x.

12. Given $3x - \frac{x-4}{4} - 4 = \frac{5x+14}{3} - \frac{1}{12}$ to find x.

13. Given $\frac{x-1}{7} + \frac{23-x}{5} = 7 - \frac{4+x}{4}$ to find x.

14. Given $\frac{7x+5}{3} - \frac{16+4x}{5} + 6 = \frac{3x+9}{2}$ to find x.

15. Given $\frac{3x+4}{5} - \frac{7x-3}{2} = \frac{x-16}{4}$ to find x.

16. Given $\frac{17-3x}{5} - \frac{4x+2}{3} = 5 - 6x + \frac{7x+14}{3}$ to find x.

17. Given $x - \frac{3x-3}{5} + 4 = \frac{20-x}{2} - \frac{6x-8}{7} + \frac{4x-4}{5}$ to find x.

18. Given $\frac{7x+16}{21} - \frac{x+8}{4x-11} = \frac{x}{3}$ to find x.

19. Given $\frac{6x+7}{9} + \frac{7x-13}{6x+3} = \frac{2x+4}{3}$ to find x.

20. Given $\frac{4x+3}{9} + \frac{7x-29}{5x-12} = \frac{8x+19}{18}$ to find x.

SOLUTION OF PROBLEMS.

(106.) The solution of a Problem by Algebra, consists of two distinct parts:

1. To express the conditions of the problem algebraically; that is, to *form* the equation.

2. To *solve* the equation.

The second part of the work has already been explained, but the first part is often more embarrassing to beginners than the second. Sometimes the statement of a problem furnishes the equation directly; and sometimes it is necessary to deduce from the statement new conditions, which are to be expressed algebraically. The former are called *explicit* conditions, and those which are deduced from them, *implicit* conditions.

It is impossible to give a general rule which will enable us to translate every problem into algebraic language. The power of doing this with facility can only be acquired by reflection and practice.

The following directions may be found of some service.

Denote the required quantities by some of the final letters of the alphabet ; then, by means of these letters with the usual signs, perform the same operations which would be necessary to verify their values if they were already known.

Problem 1. What number is that, to the double of which if 16 be added, the sum is equal to four times the required number?

Let x represent the number required.

The double of this will be $2x$.

This increased by 16 should equal $4x$.

Hence by the conditions, $2x + 16 = 4x$.

The problem is now translated into algebraic language, and it only remains to solve the equation in the usual way.

Transposing we obtain

$$16 = 4x - 2x = 2x,$$

and $$8 = x$$

or $$x = 8.$$

To verify this number, we have but to double 8, and add 16 to the result; the sum will be 32, which is equal to four times 8, according to the conditions of the problem.

Prob. 2. What number is that, the double of which exceeds its half by 6?

Let $x =$ the number required.

Then by the conditions

$$2x - \frac{x}{2} = 6.$$

Clearing of fractions

$$4x - x = 12$$

or $$3x = 12.$$

Hence $$x = 4.$$

To verify this result, double 4 which makes 8, and diminish it by the half of 4, or 2 ; the result is 6 according to the conditions of the problem.

Prob. 3. The sum of two numbers is 8, and their difference 2. What are the two numbers ?

Let $x =$ the least number.

Then $x + 2$ will be the greater number.

The sum of these is $2x + 2$, which is required to equal 8. Hence we have

$$2x + 2 = 8.$$

By transposition $2x = 8 - 2 = 6$

and $x = 3$, the least number.

Also $x + 2 = 5$, the greater number.

Verification. $\left.\begin{array}{l}5 + 3 = 8\\ 5 - 3 = 2\end{array}\right\}$ according to the conditions.

The following is a generalization of the preceding Problem.

Prob. 4. The sum of two numbers is a, and their difference b. What are the two numbers ?

Let x represent the least number

then $x + b$ will represent the greater number.

The sum of these is $2x + b$, which is required to equal a.

Hence we have

$$2x + b = a.$$

By transposition, $2x = a - b$

or $x = \dfrac{a - b}{2} = \dfrac{a}{2} - \dfrac{b}{2}$, the less number.

Hence $x + b = \dfrac{a}{2} - \dfrac{b}{2} + b = \dfrac{a}{2} + \dfrac{b}{2}$, the greater number.

As these results are independent of any particular value attributed to the letters a and b, it follows that

Half the difference of two quantities, added to half their sum, is equal to the greater ; and

Half the difference subtracted from half the sum is equal to the less.

The expressions $\frac{a}{2}+\frac{b}{2}$ and $\frac{a}{2}-\frac{b}{2}$ are called *formulas*, because they may be regarded as comprehending the solution of all questions of the *same kind;* that is, of all problems in which we have given the sum and difference of two quantities.

Thus let $\left.\begin{matrix} a=8 \\ b=2 \end{matrix}\right\}$ as in the preceding problem.

Then $\frac{a}{2}+\frac{b}{2}=\frac{8+2}{2}=5$, the greater number.

And $\frac{a}{2}-\frac{b}{2}=\frac{8-2}{2}=3$, the less number.

Given the sum of two numbers, = 10 ; their difference = 6 ; required the numbers,
12 " 2 "
23 " 11 "
100 " 50 "
100 " 1 "
5 " ½ "
10 " ½ "

Prob. 5. From two towns which are 54 miles distant, two travelers set out at the same time with an intention of meeting. One of them goes 4 miles and the other 5 miles per hour. In how many hours will they meet?

Let x represent the required number of hours.

Then $4x$ will represent the number of miles one traveled,

and $5x$ the number the other traveled;

and since they meet, they must together have traveled the whole distance.

Consequently $4x+5x=54$.

Hence $9x=54$

or $x=6$.

Proof. In 6 hours, at 4 miles an hour, one would travel 24 miles; the other at 5 miles an hour would travel 30 miles. The sum of 24 and 30 is 54 miles, which is the whole distance.

This Problem may be generalized as follows:

Prob. 6. From two places which are a miles apart, two bodies move towards each other, the one at the rate of m miles per hour, the other at the rate of n miles per hour. In how many hours will they meet?

Let x represent the required number of hours.

Then mx will represent the number of miles one body moves,

and nx the miles the other body moves,

and we shall obviously have

$$mx + nx = a.$$

Hence $x = \dfrac{a}{m+n}$.

This is a general formula, comprehending the solution of all problems of this kind. Thus

Let the distance	= 150;	one body moves 6;	the other 4	miles per hour;
	90	" 8	7	Required the time of meeting.
	135	" 15	12	
	210	" 20	15	"

We see that an infinite number of problems may be proposed, all similar to Prob. 5; but they are all solved by the formula of Prob. 6. We also see what is necessary in order that the answers may be obtained in *whole numbers*. The given distance (a) must be exactly divisible by $m + n$.

Prob. 7. A gentleman meeting three poor persons, divided 60 cents amongst them; to the second he gave twice, and to the third, three times as much as to the first. What did he give to each?

Let $x =$ the sum given to the first.

Then $2x =$ the sum given to the second.

and $3x =$ the sum given to the third.

Then by the conditions

$$x + 2x + 3x = 60.$$

That is $6x = 60$

or $x = 10.$

Therefore he gave 10, 20 and 30 cents to them respectively. The learner should verify this, and all the subsequent results.

The same Problem generalised.

Prob. 8. Divide the number a into three such parts, that the second may be m times, and the third n times as great as the first.

$$\text{Ans. } \frac{a}{1+m+n};\ \frac{ma}{1+m+n};\ \frac{na}{1+m+n}.$$

What is necessary in order that the preceding values may be expressed in whole numbers?

Prob. 9. A bookseller sold 10 books at a certain price; and afterwards 15 more at the same rate. Now at the latter time he received 25 dollars more than at the former. What did he receive for each book?

Ans. Five dollars.

The same Problem generalised.

Prob. 10. Find a number such that when multiplied successively by m and by n, the difference of the products shall be a.

$$\text{Ans. } \frac{a}{m-n}.$$

Prob. 11. A gentleman dying, bequeathed 1000 dollars to three servants. A was to have twice as much as B, and B three times as much as C. What were their respective shares?

Ans. A received \$600; B \$300; and C \$100.

Prob. 12. Divide the number a into three such parts that the second may be m times as great as the first, and the third n times as great as the second.

$$\text{Ans. } \frac{a}{1+m+mn};\ \frac{ma}{1+m+mn};\ \frac{mna}{1+m+mn}.$$

Prob. 13. A cask which held 120 gallons, was filled with a mixture of brandy, wine and water. In it there were 10 gallons of wine more than there were of brandy, and as much water as both wine and brandy. What quantity was there of each?

Ans. Brandy 25 gallons; wine 35; and water 60 gallons.

Prob. 14. Divide the number a into three such parts, that the second shall exceed the first by m; and the third shall be equal to the sum of the first and second.

Ans. $\dfrac{a-2m}{4}$; $\dfrac{a+2m}{4}$; $\dfrac{a}{2}$.

Prob. 15. A person employed four workmen; to the first of whom he gave 2 shillings more than to the second; to the second 3 shillings more than to the third; and to the third 4 shillings more than to the fourth. Their wages amount to 32 shillings. What did each receive?

Ans. They received 12, 10, 7 and 3 shillings respectively.

Prob. 16. Divide the number a into four such parts, that the second shall exceed the first by m; the third shall exceed the second by n; and the fourth shall exceed the third by p.

Ans. $\dfrac{a-3m-2n-p}{4}$; $\dfrac{a+m-2n-p}{4}$; $\dfrac{a+m+2n-p}{4}$; $\dfrac{a+m+2n+3p}{4}$.

(107.) Problems which involve several unknown quantities, may often be solved by the use of a single unknown letter. Most of the preceding examples are of this kind. In general, when we have given the sum or difference of two quantities, both of them may be expressed by means of the same letter. For the difference of two quantities added to the less must be equal to the greater; and if one of two quantities be subtracted from their sum, the remainder will be equal to the other.

Prob. 17. At a certain election 36000 votes were polled; and the candidate chosen wanted but 3000 of having twice as many votes as his opponent. How many voted for each?

Let $x =$ the number of votes for the unsuccessful candidate.

Then $36000-x=$ the number the successful one had,

And $36000-x+3000=2x$.

Ans. 13000 and 23000.

Prob. 18. Divide the number a into two such parts, that one part increased by b shall be equal to m times the other part.

Ans. $\frac{a+b}{m+1}$; $\frac{ma-b}{m+1}$.

Prob. 19. A train of cars moving at the rate of 20 miles per hour, had been gone three hours, when a second train was started at the rate of 25 miles per hour. In what time will the second train overtake the first?

Let x = the number of hours the second train is in motion,
$x+3$ = the time of the first train.
Then $25x$ = the number of miles traveled by the second train,
$20(x+3)$ = the miles traveled by the first train.

But at the time of meeting they must both have traveled the same distance.

Therefore $25x = 20x + 60$.
By transposition, $5x = 60$
and $x = 12$.

Proof. In 12 hours at 25 miles per hour, the second train goes 300 miles; and in 15 hours at 20 miles per hour, the first train also goes 300 miles; that is, is overtaken by the second train.

Prob. 20. Two bodies move in the same direction from two places at a distance of a miles apart; the one at the rate of n miles per hour, the other pursuing at the rate of m miles per hour. When will they meet?

Ans. In $\frac{a}{m-n}$ hours.

This Problem it will be seen is essentially the same as Prob. 10.

Prob. 21. Divide the number 197 into two such parts, that four times the greater may exceed five times the less by 50.

Ans. 82 and 115.

Prob. 22. Divide the number a into two such parts that m times the greater may exceed n times the less by b.

Ans. $\frac{ma - b}{m + n}$; $\frac{na + b}{m + n}$.

When $n = 1$, this Problem reduces to Problem 18.

When $b = 0$, this Problem reduces to Problem 26.

Prob. 23. What number is that whose third part exceeds its fourth part by 16?

Let $12x =$ the number.

Then $$4x - 3x = 16$$

or $$x = 16.$$

Therefore the number $= 12 \times 16 = 192$.

Prob. 24. Find a number such that when it is divided successively by m and by n, the difference of the quotients shall be a.

Ans. $\frac{mna}{m - n}$.

Prob. 25. A prize of 2329 dollars was divided between two persons, A and B, whose shares were in the ratio of 5 to 12. What was the share of each?

Beginners almost invariably put x to represent one of the quantities sought in a problem; but a solution may often be very much simplified by pursuing a different method. Thus in the preceding problem, we may put x to represent one fifth of A's share. Then $5x$ will be A's share, and $12x$ will be B's, and we shall have the equation

$$5x + 12x = 2329,$$

and $$x = 137,$$

consequently their shares were 685 and 1644 dollars.

Prob. 26. Divide the number a into two such parts, that the first part may be to the second as m to n.

Ans. $\frac{ma}{m + n}$; $\frac{na}{m + n}$.

Prob. 27. What two numbers are as 2 to 3; to each of which if four be added, the sums will be as 5 to 7?

Ans.

(108.) A rigid adherence to system would have required this example to be placed after the subject of Proportion, which is treated of in Section XIII. It is, however, only necessary to assume one simple principle which is employed in Arithmetic, viz. If four quantities are proportional, *the product of the extremes is equal to the product of the means.*

Thus if $$a : b :: c : d.$$

Then $$ad = bc.$$

In the preceding Problem, let $2x$ and $3x$ be the numbers.

Then $$2x + 4 : 3x + 4 :: 5 : 7,$$

and by the last principle

$$14x + 28 = 15x + 20.$$

Prob. 28. What two numbers are as m to n, to each of which, if a be added, the sums shall be as p to q?

Ans. $\frac{ma(p-q)}{mq-np}$; $\frac{na(p-q)}{mq-np}$.

Prob. 29. A gentleman divides a dollar among 12 children, giving to some 9 cents each, and to the rest 7 cents. How many were there of each class?

Ans.

Prob. 30. Divide the number a into two such parts, that if the first is multiplied by m, and the second by n, the sum of the products shall be b.

Ans. $\frac{b-na}{m-n}$; $\frac{ma-b}{m-n}$.

Prob. 31. If the sun moves every day one degree, and the moon thirteen, and at a certain time the sun is 60 degrees in advance of the moon; when will they be in conjunction for the first time, second time, and so on?

Prob. 32. If two bodies move in the same direction upon the circumference of a circle which measures a miles, the one at the rate of n miles per day, the other pursuing at the rate of m miles per day, when will they meet for the first time, second time, etc. supposing them to be b feet apart at starting?

Ans. In $\frac{b}{m-n}$; $\frac{a+b}{m-n}$; $\frac{2a+b}{m-n}$ etc. days.

Prob. 33. Divide the number 12 into two such parts, that the difference of their squares may be 48.

Ans.

Prob. 34. Divide the number a into two such parts that the difference of their squares may be b.

Ans. $\frac{a^2-b}{2a}$; $\frac{a^2+b}{2a}$.

Prob. 35. The estate of a bankrupt, valued at 21000 dollars, is to be divided amongst three creditors proportionably to what is due them. The debts due to A and B are as 2 to 3; while B's claims and C's are in the ratio of 4 to 5. What sum must each receive?

Ans.

Prob. 36. Divide the number a into three parts which shall be to each other as $m:n:p$.

Ans. $\frac{ma}{m+n+p}$; $\frac{na}{m+n+p}$; $\frac{pa}{m+n+p}$.

When $m = 1$, Prob. 36 reduces to the same form as Prob. 8.

Prob. 37. What sum is that whose fifth and sixth parts added together amount to 44?

Ans.

Prob. 38. Find a number such that when it is divided by m and by n, the sum of the quotients shall be a.

Ans. $\frac{mna}{m+n}$.

Prob. 39. A can perform a piece of work in 6 days; B can perform the same work in 8 days; and C can perform the same work in 24 days. In what time will they finish it if all work together?

Prob. 40. A can perform a piece of work in a days, B in b

days, and C in c days. In what time will they perform it if all work together?

Ans. $\dfrac{abc}{ab + ac + bc}$ days.

Prob. 41. There are three workmen, A, B and C. A and B together can perform a piece of work in 27 days; A and C together in 36 days; and B and C together in 54 days. In what time could they finish it if all worked together?

A and B together can perform $\frac{1}{27}$ of the work in 1 day.
A and C " $\frac{1}{36}$ " 1 "
B and C " $\frac{1}{54}$ " 1 "

Therefore adding these three results,

$$2A + 2B + 2C \text{ can perform } \tfrac{1}{27} + \tfrac{1}{36} + \tfrac{1}{54} \text{ in 1 day.}$$
$$= \tfrac{1}{12} \text{ in one day.}$$

Therefore A, B and C together can perform $\frac{1}{24}$ of the work in one day; that is they can finish it in 24 days. If we put x to represent the time in which they would all finish it, then they would together perform $\frac{1}{x}$ part of the work in one day, and we should have

$$\tfrac{1}{27} + \tfrac{1}{36} + \tfrac{1}{54} = \tfrac{2}{x}$$

Prob. 42. A and B can perform a piece of labor in a days; A and C together in b days; and B and C together in c days. In what time could they finish it if all work together?

Ans. $\dfrac{2abc}{ab + ac + bc}$ days.

This result, it will be seen, is of the same form as that of Problem 40.

Prob. 43. A has two kinds of change. It takes 20 pieces of the first to make a dollar, and 4 pieces of the second to make the same. Now B wishes to have 8 pieces for a dollar. How many of each kind must A give him?

Prob. 44. A broker has two kinds of change; there must be a pieces of the first to make a dollar, and b pieces of the second to make the same. Now a person wishes to have c pieces for a dollar. How many pieces of each kind must the broker give him?

Ans. $\frac{a(c-b)}{a-b}$ of the first kind; $\frac{b(a-c)}{a-b}$ of the second.

Prob. 45. Divide the number 45 into four such parts, that the first increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2 shall all be equal.

In solving examples of this kind, several unknown quantities are usually introduced, but this practice is worse than superfluous. The four parts into which 45 is to be divided, may be represented thus:

The first $= x - 2$,
second $= x + 2$,
third $= \frac{x}{2}$,
fourth $= 2x$,

for if the first expression be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the result in each case will be x. The sum of the four parts is $4\frac{1}{2}x$, which must equal 45.

Hence $x = 10$.

Therefore the parts are 8, 12, 5 and 20.

Prob. 46. Divide the number a into four such parts, that the first increased by m, the second diminished by m, the third multiplied by m, and the fourth divided by m shall all be equal.

Ans. $\frac{ma}{(m+1)^2} - m$; $\frac{ma}{(m+1)^2} + m$; $\frac{a}{(m+1)^2}$; $\frac{m^2a}{(m+1)^2}$.

Prob. 47. A merchant maintained himself for three years at an expense of $500 a year; and in each of those years, augmented that part of his stock which was not thus expended by one third thereof. At the end of the third year his original stock was doubled. What was that stock?

Prob. 48. A merchant supported himself for three years at an expense of a dollars per year; and each year augmented that part of his stock which was not thus expended, by one third thereof. At the end of the third year, his original stock was doubled. What was that stock?

Ans. $\frac{148a}{10}$.

Prob. 49. A father, aged 54 years, has a son aged 9 years. In how many years will the age of the father be four times that of the son?

Prob. 50. A father has lived a number of years expressed by a, his son a number of years expressed by b. In how many years will the age of the father be n times that of the son?

Ans. $\frac{a - nb}{n - 1}$.

SECTION VIII.

SOLUTION OF SIMPLE EQUATIONS, CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

(109.) In the examples which have been hitherto given, each problem has contained but one unknown quantity; or, if there have been more, they have been so related to each other that all have been expressed by means of the same letter. This, however, cannot always be done, and we are now to consider how equations of this kind are resolved.

If we have *two* equations, with two unknown quantities, we must endeavor to deduce from them *a single equation*, containing only *one* unknown quantity. We must, therefore, make one of the unknown quantities disappear, or as it is termed, we must *eliminate* it. There are three different methods of elimination which may be practised.

The *first* is by substitution,
second " comparison,
third " addition and subtraction.

ELIMINATION BY SUBSTITUTION.

(110.) Let it be proposed to solve the system of equations

$$x+y=12$$
$$x-y=\ \ 6.$$

From the second equation, we find the value of x in terms of y, which gives

$$x=y+6.$$

Substituting the expression $y+6$ for x in the first equation, it becomes

$$y+6+y=12;$$

from which we find that $y=3$; and since we have already seen that $x=y+6$, we find that $x=3+6=9$.

To verify these values, substitute them for x and y in the original equations, and we shall obtain

$$9+3=12$$
$$9-3=6.$$

Again, take the equations

$$2x+3y=13$$
$$5x+4y=22.$$

From the first equation we find

$$y=\frac{13-2x}{3}.$$

Substituting this value of y in the second equation, it becomes

$$5x+4\times\frac{13-2x}{3}=22,$$

an equation containing only x, which when solved gives

$$x=2;$$

and this value of x, substituted in either of the original equations, gives

$$y=3.$$

The method thus exemplified, is expressed in the following

RULE.

Find an expression for the value of one of the unknown quantities in one of the equations; then substitute this value in the place of its equal in the other equation.

ELIMINATION BY COMPARISON.

(111.) To illustrate this method, take the first of the preceding examples. Derive from each equation an expression for x in terms of y, we shall then have

$$x = 12 - y$$
$$x = 6 + y.$$

These two values of x must be equal to each other, and by *comparing* them we shall obtain

$$12 - y = 6 + y,$$

an equation involving only one unknown quantity;
whence $y = 3$.

Substituting this value of y in the expression $x = 6 + y$, and we find $x = 9$ as before.

Again, take the second of the preceding examples.

From equation first, we find

$$y = \frac{13 - 2x}{3}$$

and from equation second

$$y = \frac{22 - 5x}{4}.$$

Putting these values of y equal to each other, we have

$$\frac{13 - 2x}{3} = \frac{22 - 5x}{4},$$

an equation containing only x, whence we obtain

$$x = 2.$$

Substituting this value of x in either of the preceding expressions for y, we find

$$y = 3.$$

The method thus exemplified is expressed in the following

RULE.

Find an expression for the value of one of the unknown quantities in each of the equations, and form a new equation by putting one of these values equal to the other.

ELIMINATION BY ADDITION AND SUBTRACTION.

(112.) To illustrate this method, take the first of the same two examples as in Art. 110. Since the coefficients of y in the two equations are equal and have contrary signs, we may eliminate this letter by adding the two equations together, whence we obtain

$$2x = 18$$
$$\text{or } x = 9.$$

We may now deduce the value of y by substituting the value of x in one of the original equations. Taking the first for example, we have

$$9 + y = 12,$$

whence $y = 3.$

Since the coefficients of x are equal in the two original equations, we might have eliminated this letter by subtracting one equation from the other. Subtracting the first from the second we obtain

$$2y = 6$$
$$\text{or } y = 3.$$

Let us apply the same method to the second example of Art. 110. We perceive that if we could deduce from the proposed equations two other equations, in which the coefficients of y should be equal, the elimination of y might be effected by *subtracting* one of these new equations from the other.

It is easily seen that we shall obtain two equations of the form required, if we multiply all the terms of each equation by the coefficient of y in the other. Multiplying therefore, all the terms of equation first by 4, and all the terms of equation second by 3, they become

$$8x + 12y = 52$$
$$15x + 12y = 66.$$

Subtracting the former of these equations from the latter, we find

$$7x = 14,$$

whence $x = 2.$

In like manner, in order to eliminate x, multiply the first of the proposed equations by 5, and the second by 2, they will become

$$10x + 15y = 65$$
$$10x + 8y = 44.$$

Subtracting the latter of these two equations from the former, we have

$$7y = 21,$$

whence $$y = 3.$$

This last method is expressed in the following

RULE.

Multiply or divide the equations, if necessary, in such a manner that one of the unknown quantities shall have the same coefficient in both. Then subtract one equation from the other, if the signs of these coefficients are the same, or add them together if the signs are different.

EXAMPLES.

(113.) 1. Given $\left.\begin{array}{r}5x + 4y = 58 \\ 3x + 7y = 67\end{array}\right\}$ to find the values of x and y.

By the first method.

From the second equation we find

$$3x = 67 - 7y.$$

Therefore $x = \dfrac{67 - 7y}{3}$.

Substituting this value of x in the first equation,

$$5 \times \frac{67 - 7y}{3} + 4y = 58.$$

Hence $335 - 35y + 12y = 174.$

By transposition, $335 - 174 = 35y - 12y,$

or $161 = 23y.$

Therefore $y = 7.$

Substituting this value of y in the expression for the value of x given above, it becomes

$$\frac{67-7\times 7}{3}=\frac{67-49}{3}=\frac{18}{3}=6.$$

Thus we have $y=7,\ x=6.$

By the second method.

From the first equation we find

$$5x=58-4y,$$

whence $x=\frac{58-4y}{5}.$

From the second equation $x=\frac{67-7y}{3}$

Therefore $\frac{58-4y}{5}=\frac{67-7y}{3}.$

Clearing of fractions, $174-12y=335-35y.$

By transposition, $35y-12y=335-174,$

or $23y=161.$

Therefore $y=7,$

whence as before $x=6.$

By the third method.

Multiplying the second equation by 5 and the first by 3, we obtain

$$15x+35y=335$$

and $15x+12y=174.$

By subtraction $23y=161,$

or $y=7.$

Whence from equation first,

$$5x=58-4y=58-28=30,$$

and therefore $x=6.$

(114.) Thus the same example may be solved by either of the three methods, and each method has its advantages in particular cases. Generally, however, the first two methods give rise to fractional expressions which occasion inconvenience in practice, while the third method is not liable to this objection. When the coefficient of one of the unknown quantities in one of the equations is equal to unity, this inconvenience does not occur, and the method by substitution may be preferable; the third will, however, commonly, be found most convenient.

Example 2. Given $\left.\begin{aligned}11x+3y&=100\\ 4x-7y&=4\end{aligned}\right\}$ to find the values of x and y.

Multiplying the first equation by 7 and the second by 3 we obtain

$$77x+21y=700$$
$$12x-21y=12.$$

Therefore by addition $89x=712$

$$\text{or } x=8.$$

From equation first, $3y=100-11x$

$$=100-88=12$$

and $y=4$.

These values of x and y may be easily verified by substitution in the original equations.

Thus $11\times8+3\times4=100$; or $88+12=100$.

And $4\times8-7\times4=4$; or $32-28=4$.

Ex. 3. Given $\left.\begin{aligned}\frac{x}{2}+\frac{y}{3}&=7\\ \frac{x}{3}+\frac{y}{2}&=8\end{aligned}\right\}$ to find the values of x and y.

Ans. $x=6$, $y=12$.

Ex. 4. Given $\left.\begin{aligned}\frac{x+2}{3}+8y&=31\\ \frac{y+5}{4}+10x&=192\end{aligned}\right\}$ to find x and y.

Ex. 5. Given $\left.\begin{aligned}2y-\frac{x+3}{4}&=7+\frac{3x-2y}{5}\\ 4x-\frac{8-y}{3}&=24\tfrac{1}{2}-\frac{2x+1}{2}\end{aligned}\right\}$ to find x and y.

Ex. 6. Given $\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}&=m\\ \frac{c}{x}+\frac{d}{y}&=n\end{aligned}\right\}$ to find x and y.

$$\text{Ans. } x=\frac{bc-ad}{nb-md}$$

$$y=\frac{bc-ad}{mc-na}.$$

(115.) When a problem involves a large number of quantities, it is common to designate a part of them by different letters, and for the remaining quantities to employ the same letters accented or numbered.

Thus $a,\ a',\ a'',\ a''',\ a^{\text{IV}},\ \text{- - - - -}\ a^{(m)}$
$a^{(1)},\ a^{(2)},\ a^{(3)},\ a^{(4)},\ \text{- - - - -}\ a^{(m)}$
$a_1,\ a_2,\ a_3,\ a_4,\ \text{- - - - -}\ a_{(m)}$

are used to denote different quantities, though they generally imply some similarity between the quantities which they represent. a' is read *a prime; a'', a second; a''', a third*, etc. We must carefully distinguish between a_2 and a^2; between a_4 and a^4, etc. In the one case the numerals are exponents, and denote powers of *a;* while in the other case, the numerals are only used for the sake of convenience to denote distinct quantities.

Ex. 8. Given $\left.\begin{aligned}ax+by&=c\\ a'x+b'y&=c'\end{aligned}\right\}$ to find the values of x and y.

$$\text{Ans. } x=\frac{b'c-bc'}{ab'-a'b};\ y=\frac{ac'-a'c}{ab'-a'b}.$$

The symmetry of these expressions is calculated to fix them in the memory.

Ex. 8. What fraction is that, to the numerator of which if 4 be added, the value is one half; but if 7 be added to the denominator, its value is one fifth?

Let $\frac{x}{y}$ represent the fraction required.

Then by the first condition

$$\frac{x+4}{y} = \tfrac{1}{2}; \text{ whence } 2x+8=y.$$

By the second condition

$$\frac{x}{y+7} = \tfrac{1}{5}; \text{ whence } 5x=y+7.$$

Subtracting the first equation from the second,

$$3x-8=7,$$

whence

$$3x=15$$

$$\text{or } x=5.$$

Therefore $y=2x+8=10+8=18$,
and the fraction is $\frac{5}{18}$.

Proof.

$$\frac{5+4}{18}=\frac{1}{2},$$

and

$$\frac{5}{18+7}=\frac{1}{5}.$$

Ex. 9. What fraction is that, to the numerator of which if a be added, the value is m; but if b be added to the denominator, its value is n?

Ex. 10. A certain sum of money, put out at simple interest, amounts in 8 months to $1488; and in 15 months, it amounts to $1530. What is the sum and the rate per cent?

Ex. 11. A sum of money put out at simple interest amounts in m months to a dollars; and in n months to b dollars.

Required the sum and rate per cent?

The sum is $\dfrac{na-mb}{n-m}$; the rate is $1200 \times \dfrac{b-a}{na-mb}$.

Ex. 12. There is a number consisting of two digits, the second of which is greater than the first; and if the number be divided by the sum of its digits, the quotient is 4; but if the digits be inverted, and that number be divided by a number greater by two than the difference of the digits, the quotient is 14.

Required the number.

Let x represent the left hand digit
and y " right hand "

Then since x stands in the place of tens, the number will be represented by $10x + y$.

Hence by the first condition,

$$\frac{10x + y}{x + y} = 4.$$

By the second condition,

$$\frac{10y + x}{y - x + 2} = 14.$$

Whence $x = 4, y = 8,$
and the required number is 48.

Ex. 13. What fraction is that whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$?

Ex. 14. A person expends thirty pence in apples and pears, buying his apples at 4 and his pears at 5 for a penny; and afterwards accommodates his neighbor with half his apples and one third of his pears for 13 pence. How many did he buy of each?

EQUATIONS WHICH CONTAIN THREE OR MORE UNKNOWN QUANTITIES.

(116.) Let us now consider the case of *three* equations involving *three* unknown quantities.

Take the system of equations

$$3x + 2y + z = 16 \qquad (1)$$
$$2x + 2y + 2z = 18 \qquad (2)$$
$$2x + 3y + z = 17. \qquad (3)$$

In order to eliminate z between equations (1) and (2), we will divide both members of the second equation by two; we thus obtain

$$x + y + z = 9.$$

Subtracting this from the first equation, we find a new equation between two unknown quantities,

$$2x + y = 7. \qquad (\alpha)$$

In order to eliminate z between equations (1) and (3), subtract the former from the latter, which gives

$$-x + y = 1. \qquad (\beta)$$

From the two equations, (α) and (β), one may be deduced containing only one unknown quantity. For by subtracting one from the other we have

$$3x = 6, \text{ or } x = 2.$$

Substituting this value of x in equation (β) we obtain

$$y = 3.$$

Substituting these values of x and y in equation (1), we obtain

$$3 \times 2 + 2 \times 3 + z = 16.$$

Hence $$z = 4.$$

These values of x, y and z may be verified by substitution in the original equations.

We have effected the elimination in this case by method third, Art. 112; but either of the other methods might have been employed. Hence to solve three equations containing three unknown quantities, we have the following

RULE.

(117.) *From the three equations, deduce two containing only two unknown quantities; then from these two deduce one containing only one unknown quantity.*

Ex. 15. Given $$\left.\begin{array}{r} x + y + z = 29 \ (1) \\ x + 2y + 3z = 62 \ (2) \\ \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 10 \ (3) \end{array}\right\} \text{to find } x, y \text{ and } z.$$

Subtract equation (1) from (2) and we obtain

$$y + 2z = 33 \qquad (\alpha)$$

clearing equation (3) of fractions

$$6x + 4y + 3z = 120. \qquad (4)$$

Multiplying equation (1) by 6,

$$6x + 6y + 6z = 174. \qquad (5)$$

Subtracting (4) from (5), $2y + 3z = 54$ (β).

We have thus obtained two equations (α) and (β), containing two unknown quantities.

Multiplying (α) by 2 we have $2y + 4z = 66$. (6)

Subtracting (β) from (6) $z = 12$.

Substituting this value of z in (β)

$$2y + 36 = 54.$$

Whence $y = 9$.

Substituting these values of y and z in equation (1)

$$x + 9 + 12 = 29.$$

Whence $x = 8$.

These values may be verified as in former examples.

Ex. 16. Given $\left.\begin{array}{r} 2x + 4y - 3z = 22 \\ 4x - 2y + 5z = 18 \\ 6x + 7y - z = 63 \end{array}\right\}$ to find x, y and z.

Ex. 17. Given $\left.\begin{array}{r} x + y = a \\ x + z = b \\ y + z = c \end{array}\right\}$ to find x, y and z.

Ex. 18. Given $\left.\begin{array}{r} x + \frac{1}{2}y + \frac{1}{3}z = 32 \\ \frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 15 \\ \frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z = 12 \end{array}\right\}$ to find x, y and z.

(118.) If we had *four* equations involving *four* unknown quantities, we might by the methods already explained eliminate one of the unknown quantities. We should thus obtain *three* equations between *three* unknown quantities, which might be solved according to Art. 116. So also if we had *five* equations involving *five* unknown quantities, we might by the same process reduce them to *four* equations involving *four* unknown quantities; then to *three*, and so on. By following the same method, we might resolve a system of any number of equations of the first degree. Hence if we have m equations, involving m unknown quantities, the unknown quantities may be eliminated by the following

RULE.

1. *Combine successively any one of the equations with each of the* $m-1$ *others, so as to eliminate the same unknown quantity; you will thus obtain* $m-1$ *new equations containing* $m-1$ *unknown quantities.*

2. *Eliminate a new unknown quantity by combining one of these new equations with the* $m-2$ *others; this will give* $m-2$ *equations containing* $m-2$ *unknown quantities.*

3. *Continue this series of operations until you arrive at a single equation containing but one unknown quantity, from which the value of this unknown quantity is easily deduced. Then by going back step by step to one of the original equations, the values of the other unknown quantities may be successively determined.*

Ex. 19. Given

$$\left.\begin{array}{r} 7x-2z+3u=17 \\ 4y-2z+t=11 \\ 5y-3x-2u=8 \\ 4y-3u+2t=9 \\ 3z+8u=33 \end{array}\right\} \text{ to find } x, y, z, u \text{ and } t.$$

Ans. $x=2$, $y=4$, $z=3$, $u=3$, $t=1$.

(119.) Hitherto we have supposed the number of equations equal to the number of symbols employed to denote the unknown quantities. This must be the case with every problem in order that it may be *determinate;* that is, that it may not admit of an indefinite number of solutions.

Suppose for example, that a problem involving two unknown quantities, (x and y) leads to the single equation,

$$x-y=3.$$

Now if we make $y=1$ then $x=4$
$y=2$ " $x=5$
$y=3$ " $x=6$
$y=4$ " $x=7$
etc. etc.

and each of these systems of values, 1 and 4; 2 and 5; 3 and 6, etc. substituted for x and y in the original equation, will satisfy it equally well. Hence the problem is *indeterminate;* that is, admits of an indefinite number of solutions.

(120.) If we had *two* equations involving *three* unknown quantities, we could in the first place eliminate one of the unknown quantities by means of the proposed equations, and thus obtain *one* equation containing *two* unknown quantities, which would be satisfied by an infinite number of systems of values. Therefore in order that a problem may be *determinate*, its enunciation must contain as many different conditions as there are unknown quantities, and each of these conditions must be expressed by an *independent* equation.

Equations are said to be *independent*, when they express conditions *essentially different;* and *dependent*, when they express the *same* conditions under different forms.

Thus $\left.\begin{aligned} x+y&=7\\ 2x+y&=10\end{aligned}\right\}$ are *independent* equations.

But $\left.\begin{aligned} x+y&=7\\ 2x+2y&=14\end{aligned}\right\}$ are *not* independent,

because the one may be deduced from the other.

(121.) If on the contrary, the number of *independent* equations exceeds the number of unknown quantities, these equations will be *contradictory*.

For example, let it be required to find two numbers such that their sum shall be 7, their difference 1, and their product 100.

From these conditions we derive the following equations:

$$x+y=7$$
$$x-y=1$$
$$xy=100.$$

From the first two equations we easily find

$$x=4, \text{ and } y=3.$$

Hence the third condition, which requires that their product shall be equal to 100, *cannot be fulfilled.*

Ex. 20. A person owes a certain sum to two creditors. At one time he pays them \$530, giving to one four elevenths of the sum which is due; and to the other \$30 more than one sixth of his debt to him. At a second time, he pays them \$420, giving to the first, three sevenths of what remains due to him; and to the other, one third of what remains due to him. What were the debts?

Ex. 21. If A and B together can perform a piece of work in 12 days, A and C together in 15 days, and B and C in 20 days; how many days will it take each person to perform the same work alone?

Ex. 22. If A and B together can perform a piece of work in a days, A and C together in b days, and B and C in c days, how many days will it take each person to perform the same work alone?

SECTION IX.

DISCUSSION OF EQUATIONS OF THE FIRST DEGREE.

(122.) By the *discussion of an equation* we understand, the determination of the values which the unknown quantities assume for particular hypotheses made upon the values of the quantities which are given, and the interpretation of the peculiar results obtained. The term, therefore, is not strictly applicable, except to problems which are stated in the most general form, like some of those in Arts. 106 to 108. If the sum of two numbers is represented by a and their difference by b, the greater number will be expressed by $\frac{a}{2}+\frac{b}{2}$; and the less by $\frac{a}{2}-\frac{b}{2}$. Here a and b may have any value whatever, and still these formulæ will always hold true. It frequently happens that by attributing different values to the letters which represent known quantities, the value of the unknown quantities assume peculiar forms which deserve consideration.

(123.) We may obtain five species of values for the unknown quantity in a problem of the first degree.

I. Positive values.

II. Negative values.

III. Values of the form of 0, or $\frac{0}{A}$.

IV. Values of the form of $\frac{A}{0}$.

V. Values of the form of $\frac{0}{0}$.

We will consider these five cases in succession.

I. *Positive values* are generally answers to problems in the sense in which they are proposed. Nevertheless all positive values will not always satisfy the enunciation of a problem. If, for example, a problem requires an answer in whole numbers, and we obtain a fractional value, the problem is impossible. Thus in Problem 17, page 85, it is implied that the value of x must be a whole number, although this condition is not expressed in any equation. It would be easy to change the data of the problem so as to obtain a fractional value of x, which would indicate an impossibility in the problem proposed. Problem 43, page 90, is of the same kind.

If the value obtained for the unknown quantity even when positive does not satisfy all the conditions of the problem, the problem is impossible in the form proposed.

(124.) II. *Negative values.*

Let it be proposed to find a number, which added to the number b, gives for a sum the number a.

Let $x=$ the required number.

Then by the terms of the problem

$$x+b=a, \text{ whence } x=a-b.$$

This formula will give the value of x for every case of the proposed problem.

For example, let $a=7,\ b=4$.
Then $x=7-4=3$.
Again let $a=5,\ b=8$.
Then $x=5-8=-3$.

We thus obtain for x a *negative* value. How is it to be interpreted?

By referring to the problem, we see that it is proposed to find a number which added to 8 shall make it equal to 5. Considered arithmetically, the problem is plainly impossible. Nevertheless, if in the equation of the problem $8+x=5$, we substitute for $+x$ its value -3, it becomes

$$8-3=5,$$

an identical equation; that is, 8 diminished by 3 is equal to 5.

The negative solution $x=-3$, shows therefore the impossibility of satisfying the enunciation of the problem as above stated; but taking this value of x with a contrary sign, we see that it satisfies the enunciation when modified as follows:

To find a number which *subtracted* from 8 gives a *difference* of 5; an enunciation which differs from the former only in this that we put *subtract* for *add*, and *difference* for *sum*.

If we wish to solve this new question directly, we have but to form the equation

$$8-x=5.$$

Whence $x=8-5$, or $x=3$.

(125.) For another example, take Problem 50, page 92. The age of the father being represented by a, that of the son by b, then $\frac{a-nb}{n-1}$ will represent the number of years before the age of the father will be n times that of the son.

Thus suppose $a=54$, $b=9$ and $n=4$.

Then $$x=\frac{54-36}{3}=\frac{18}{3}=6.$$

That is, the father having lived 54 years and the son 9, in 6 years more, the father will be 60 years old, and the son 15. But 60 is 4 times 15; hence this value $x=6$, satisfies the enunciation of the problem.

Again, suppose $a=45$, $b=15$ and $n=4$.

Then $$x=\frac{45-60}{3}=\frac{-15}{3}=-5.$$

Here again we obtain a *negative solution*. How are we to interpret it?

By referring to the problem, we see that the age of the son is already *more* than one fourth that of the father, so that the time required is already *past* by five years. The value of x just obtained, taken with a contrary sign, satisfies the following enunciation:

A father is 45 years old, his son 15; *how many years since* the age of the father was four times that of his son?

The equation of this new enunciation would be

$$15 - x = \frac{45 - x}{4}.$$

Whence $60 - 4x = 45 - x$; and $x = 5$.

(126.) Reasoning from analogy, we are led to the following general principle:

1. *Every negative value found for the unknown quantity in a problem of the first degree, indicates an absurdity in the conditions of the enunciation, or at least in its algebraic statement.*

2. *This value taken with a contrary sign, may be regarded as the answer to a problem, of which the enunciation only differs from that of the proposed problem in this, that certain quantities which were* ADDED *should have been* SUBTRACTED, *and reciprocally.*

(127.) In what case would the value of the unknown quantity in Prob. 20, page 86, be negative?

Ans. when $n > m$.

Thus let $m = 20$, $n = 25$, and $a = 60$ miles.

Then $$x = \frac{60}{20 - 25} = \frac{60}{-5} = -12.$$

To interpret this result, observe that it is impossible that the second train which moves the slowest, should overtake the first. At the time of starting, the distance between them was 60 miles, and every subsequent hour the distance increases. If, however, we suppose the two trains to *have been* moving uniformly along an endless road, it is obvious that *at some former time, they must have been together.*

This negative solution then shows an absurdity in the conditions of the problem. The problem should have been stated thus:

Two trains of cars, 60 miles apart, are moving in the same direction, the forward one 25 miles per hour, the other 20. *How long since they were together?*

Let $x =$ the required number of hours.

Then $25x =$ the distance traveled by the first train,

$20x =$ " " second "

And as they are now 60 miles apart

$$25x = 20x + 60.$$

Hence $$5x = 60,$$

and $$x = +12.$$

We thus obtain a positive value of x.

In order to include both of these cases in the same enunciation, the question should have been asked: *Required the time of their being together*, leaving it uncertain whether the time was *past* or *future*.

In what case would the value of one of the unknown quantities in Problem 34, page 89 be negative? Why should it be negative; and how could the enunciation be corrected for this case?

(128.) III. *Values of the form of* 0 *or* $\frac{0}{A}$.

In what case would the value of the unknown quantity in Problem 20, page 86, become 0, and what would this value signify?

Ans. This value becomes zero when $a = 0$, which signifies that the two trains are together at the outset.

In what case would the value of the unknown quantity in Problem 50, page 92, become zero, and what would this value signify?

Ans. when $a = nb$, which signifies that the age of the father is *now* n times that of the son.

When a problem gives zero for the value of the unknown quantity, this value is sometimes a true solution, and sometimes it indicates an impossibility in the proposed question.

(129.) IV. *Values of the form of* $\frac{A}{0}$.

In what case does the value of the unknown quantity in Problem 20, page 86 reduce to $\frac{A}{0}$? and how shall we interpret this result?

Ans. When $m = n$.

On referring to the enunciation of the problem, we see that it is absolutely impossible to satisfy it; that is, there can be no point of meeting, for the two trains being separated by the distance a, and moving equally fast, will always continue at the same distance from each other. The result $\frac{a}{0}$, may then be regarded as a sign of *impossibility.*

Nevertheless algebraists regard the result $\frac{a}{0}$, as forming a species of value to which they have given the name of *infinity;* and for the following reason:

When the difference $m - n$, without being absolutely nothing, is *very small*, the result $\frac{a}{m-n}$ is *very large.*

For example, let $m - n = 0.01$

Then $$x = \frac{a}{m-n} = \frac{a}{.01} = 100a.$$

Let $$m - n = 0.0001$$

$$\frac{a}{m-n} = \frac{a}{.0001} = 10000a.$$

Hence if the difference in the rates of motion is not zero, the two trains must meet; and the time will become greater and greater as this difference is diminished. *If then we suppose this difference less than any assignable quantity, the time represented by* $\frac{a}{m-n}$ *will be greater than any assignable quantity, or infinite.*

Hence we infer, that every expression of the form $\frac{A}{0}$, found for the unknown quantity, indicates the impossibility of satisfying the problem, at least in *finite* numbers.

In what case would the values of the unknown quantities in Problem 10, page 84, reduce to the form $\frac{A}{0}$? and how shall we interpret this result?

(130.) V. *Values of the form of* $\frac{0}{0}$.

In what case does the value of the unknown quantity in Problem 20, page 86, reduce to $\frac{0}{0}$? and how shall we interpret this result?

Ans. When $m = n$ and $a = 0$.

To interpret this result, let us recur to the enunciation and observe that since a is zero, both trains start from the same point; and since they both travel at the same rate, *they will always remain together*, and therefore the required point of meeting will be any where in the road traveled over. The problem then is entirely *indeterminate*, or admits of an infinite number of solutions, and the expression $\frac{0}{0}$ is the symbol of an *indeterminate quantity*.

We infer, therefore, that an expression of the form $\frac{0}{0}$ found for the unknown quantity, generally indicates that it may have any value whatever. In some cases, however, this value is subject to limitations.

In what case would the values of the unknown quantities in Problem 44, page 90, reduce to $\frac{0}{0}$? and how would they satisfy the conditions of the problem?

Ans. When $a = b = c$,

which indicates that the coins are all of the same value. B might therefore be paid in either kind of coin; but there is a limitation, viz. that the value of the coins must be one dollar.

OF INFINITY.

(131.) From Art. 129, it is seen that in Algebra we sometimes have occasion to consider infinite quantities. It is necessary, therefore, to establish some general principles respecting them.

An infinite quantity is one which exceeds any assignable limit. It may be expressed by the character ∞. Thus a line produced beyond any assignable limit, is said to be of infinite length. A surface indefinitely extended, and also a solid of indefinite extent in any one of its three dimensions, are examples of infinity.

An infinite quantity does not mean an infinite *number of terms.* Thus the fraction $\frac{1}{3}$ reduced to a decimal, is .3 3 3 3 3 3 etc. without end, but the value of this series is less than unity.

Infinite quantities are *not all equal* among themselves.

Thus the series $1 + 1 + 1 + 1 + 1 +$ etc.

$2 + 2 + 2 + 2 + 2 +$ etc.

$3 + 3 + 3 + 3 + 3 +$ etc.

continued to an infinite number of terms will each be infinite, although the second series will be double, and the third treble the first.

So also a line may be infinitely extended both ways; or it may be infinitely extended in one direction and limited in the other. In either case the line is said to be infinite.

By nothing, is to be understood a quantity less than any assignable quantity.

Thus take the series of fractions $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, $\frac{1}{10000}$ etc. By increasing the denominator, we diminish the value of the fraction; and if the denominator be made infinitely great, the quotient will be zero.

We have seen in Art. 129, that $\frac{a}{0} = \infty$, where a may represent any finite quantity. That is

If a finite quantity be divided by zero, the quotient is infinite.

From the same equation we deduce $\frac{a}{\infty} = 0$. That is

If a finite quantity be divided by infinity, the quotient is zero.

From the same equation we deduce $a = 0 \times \infty$. That is
If zero be multiplied by infinity, the product is finite.

If a finite quantity be multiplied by a proper fraction, it will be *diminished*, and the smaller the multiplier, the less the product. Hence if the multiplier be infinitely small, the product will be infinitely small, or $a \times 0 = 0$. That is
If a finite quantity be multiplied by zero, the product will be zero.

From this equation, we deduce $a = \frac{0}{0}$; that is

If zero be divided by zero, the quotient is a finite quantity.

The greater the multiplier, the greater will be the product. Hence, *if a finite quantity be multiplied by infinity, the product will be infinite;* that is

$$a \times \infty = \infty.$$

From this equation we deduce $a = \frac{\infty}{\infty}$; that is
If infinity be divided by infinity, the quotient is finite.

An infinite quantity cannot be increased by the addition of a finite quantity, or diminished by its subtraction; that is $\infty \pm a = \infty$.

So also, a finite quantity is not altered by the addition or subtraction of nothing; that is $a \pm 0 = a$.

It has been shown, Art. 70, that $a^0 = 1$; that is *any* quantity with the exponent 0 is equal to 1, hence the same must be true when this quantity is infinitely small; that is $0^0 = 1$.

OF INEQUALITIES.

(132.) In discussing algebraical problems, as shown in Arts. 122—130, it is frequently necessary to employ *inequalities*, or expressions of two quantities which are *not* equal to each other. Generally the principles already established for the transformation of equations are applicable to inequalities also. There are, however, some important exceptions to be noted, arising chiefly from the use of *negative expressions* as *quantities*.

Two inequalities are said to subsist in *the same sense*, when the greater quantity stands at the left in both, or at the right in both; and in *a contrary sense*, when the greater quantity stands at the right in one, and at the left in the other.

Thus $9 > 7$ and $7 > 6$.

As also $5 < 8$ and $3 < 4$,

are inequalities which subsist in the same sense; and the inequalities $10 > 6$ and $3 < 7$,

subsist in a contrary sense.

(133.) I. *If we add the same quantity to both members of an inequality, or subtract the same quantity from both members; the resulting inequality will always subsist in the same sense.*

Thus $8 > 3$.

Adding 5 to both members,

$$8 + 5 > 3 + 5,$$

and subtracting 5 from both members,

$$8 - 5 > 3 - 5.$$

So also $3 < -2$.

Adding 6 to both members,

$$-3 + 6 < -2 + 6, \text{ or } 3 < 4;$$

and subtracting 6 from both members,

$$-3 - 6 < -2 - 6, \text{ or } -9 < -8.$$

The student must here bear in mind what was stated in Art. 47, of two negative quantities, that is the *least* whose numerical value is the *greatest*.

This principle enables us to *transpose* any term from one member of an inequality to the other by changing its sign.

Thus $a^2 + b^2 > 3b^2 - 2a^2$.

Adding $2a^2$ to each member of the inequality, it becomes

$$a^2 + b^2 + 2a^2 > 3b^2.$$

Subtracting b^2 from each member,

$$a^2 + 2a^2 > 3b^2 - b^2,$$

$$\text{or } 3a^2 > 2b^2.$$

II. *If we add together the corresponding members of two or more inequalities which subsist in the same sense, the resulting inequality will always subsist in the same sense.*

Thus $5 > 4$

$4 > 2$

$7 > 3$

Adding, $16 > 9$.

III. *But if we subtract the corresponding members of two or more inequalities which subsist in the same sense, the resulting inequality will* NOT ALWAYS *subsist in the same sense.*

Take the inequalities $4 < 7$

$2 < 3$

Subtracting, we have still $4 - 2 < 7 - 3$, or $2 < 4$.

But take $9 < 10$

and $6 < 8$

Subtracting, the result is $9 - 6 >$ (not $<$) $10 - 8$; or $3 > 2$.

We must therefore avoid as much as possible the use of this transformation; or when we employ it, determine in what sense the resulting inequality subsists.

IV. *If we multiply or divide the two members of an inequality by a positive number, the resulting inequality will subsist in the same sense.*

Thus if $a < b$.

Then $ma < mb$.

And $\frac{a}{m} < \frac{b}{m}$.

If $-a > -b$.

Then $-na > -nb$.

And $$-\frac{a}{n} > -\frac{b}{n}.$$

This principle will enable us to clear an inequality of fractions. Thus if we have

$$\frac{a^2-b^2}{2d} > \frac{c^2-d^2}{3a}.$$

Multiplying both members by $6ad$, it becomes

$$3a(a^2-b^2) > 2d(c^2-d^2).$$

V. *If we multiply or divide the two members of an inequality by a negative number, the resulting inequality will subsist in a contrary sense.*

Take for example $$8 > 7.$$

Multiplying both members by -3, we have the opposite inequality.

$$-24 < -21.$$

So also $$8 > 7.$$

But $$\frac{8}{-3} < \frac{7}{-3} \text{ or } -\frac{8}{3} < -\frac{7}{3}.$$

Therefore if we multiply or divide the two members of an inequality by a number expressed algebraically, it is necessary to ascertain whether the multiplier or divisor is negative, for in this case the inequality subsists in a contrary sense.

VI. *We cannot change the signs of both members of an inequality, unless we reverse the sense of the inequality,* for this transformation is evidently the same as multiplying both members by -1.

VII. *If both members of an inequality are positive numbers, we can raise them to any power without altering the sense of the inequality.*

Thus $$5 > 3,$$
hence, $$5^2 > 3^2, \text{ or } 25 > 9.$$

So also if $a > b$

$a^n > b^n$.

VIII. *If both members of an inequality are not positive numbers, and they be raised to any power, the resulting inequality will not always subsist in the same sense.*

Thus $-2 < +3$.

Gives $(-2)^2 < 3^2$, or $4 < 9$.

But $-3 > -5$.

Gives $(-3)^2 < (-5)^2$, or $9 < 25$.

IX. *In extracting the root of both members of an inequality it is sometimes necessary to reverse the sense of the inequality.*

Thus from $9 < 25$.

By extracting the square root we obtain either $3 < 5$,

or $-3 > -5$.

EXAMPLES.

1. Given $7x - 3 < 25$, to find the limit of x.

Ans. $x < 4$.

2. Given $2x + \frac{x}{3} - 8 < 6$ to find the limit of x.

Ans. $x < 6$.

3. Given $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{6} + \frac{x}{12} - 7 > 9$, to find the limit of x.

4. Given $\left.\begin{array}{r} \frac{bx}{2} + cx - ac < \frac{ab}{2} \\ ex + \frac{dx - bd}{3} > be \end{array}\right\}$, to find the limits of x.

5. A man being asked how much he gave for his watch, replied; If you multiply the price by 4 and to the product add 60, the sum will exceed 256; but if you multiply the price by 3,

and from the product subtract 40, the remainder will be less than 113. Required the price of the watch.

6. What sum of money is that whose half and third part added together are less than 105; but its half diminished by its fifth part is greater than 33?

7. The double of a number diminished by 6 is greater than 24; and triple the number diminished by 6 is less than double the number increased by 10. Required the number.

SECTION X.

INVOLUTION AND POWERS.

(134.) According to Art. 21, *the products formed by the successive multiplication of the same number by itself, are called the powers of that number.*

Thus, the first power of 3 is 3.

The second power of 3 is 9 or 3 × 3.

The third power of 3 is 27 or 3× 3 × 3.

The fourth power of 3 is 81 or 3 × 3 × 3 × 3,

etc. etc. etc.

(135.) According to Art. 20, *the exponent or index is a number or letter written a little above a quantity to the right, and denotes the number of times that quantity enters as a factor into a product.*

Thus the first power of a is a^1; where the exponent is 1, which, however, is commonly omitted.

The second power of a is $a \times a$ or a^2; where the exponent 2 denotes that a is taken twice as a factor to produce the power aa.

The third power of a is $a \times a \times a$, or a^3; where the exponent 3 denotes that a is taken three times as a factor to produce the power aaa.

The fourth power of a is $a \times a \times a \times a$, or a^4.

Also, the nth power of a is $a \times a \times a \times a$ . . . repeated as a factor n times, and is written a^n.

Indices may be applied to polynomials as well as to monomials.

Thus $(a + b + c)^3$ is the same as

$$(a + b + c) \times (a + b + c) \times (a + b + c),$$

or the third power of the entire expression $a + b + c$.

(136.) According to the rule for the multiplication of monomials, Arts. 49 and 50,

$$(3ab^2)^2 = 3ab^2 \times 3ab^2 = 9a^2b^4.$$

So also $(4a^2bc^3)^2 = 4a^2bc^3 \times 4a^2bc^3 = 16a^4b^2c^6$.

Hence it appears that *in order to square a monomial, we must square its coefficient, and multiply the exponent of each of the letters by 2.*

EXAMPLES.

1. Required the square of $7axy$.

Ans. $49a^2x^2y^2$.

2. Required the square of $11a^3bcd^2$.
3. Required the square of $12a^2xy$.
4. Required the square of $15ab^2cx^4$.
5. Required the square of $18x^2yz^3$.

(137.) According to Art. 53, + multiplied by +, and — multiplied by — give +. Now the square of a monomial being the product of the monomial by itself, it necessarily follows that *whatever may be the sign of a monomial, its square must be affected* with the sign +.

Thus the square of $+ 3ax$ or of $- 3ax$, is $+ 9a^2x^2$.

(138.) The method of involving a quantity to *any* power, is easily derived from the preceding principles.

Let it be required to form the fifth power of $2a^3b^2$.

According to the rules for multiplication,

$$\begin{aligned}(2a^3b^2)^5 &= 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2 \\ &= 32a^{15}b^{10}.\end{aligned}$$

Where we perceive

1. That the coefficient has been raised to the fifth power.

2. That the exponent of each of the letters has been multiplied by 5.

In like manner

$$\begin{aligned}(3a^2b^3c)^3 &= 3a^2b^3c \times 3a^2b^3c \times 3a^2b^3c\\ &= 3^3a^{2+2+2}b^{3+3+3}c^{1+1+1}\\ &= 27a^6b^9c^3.\end{aligned}$$

Hence we deduce the following general

RULE TO RAISE A MONOMIAL TO ANY POWER.

Raise the numerical coefficient to the given power, and multiply the exponent of each of the letters by the index of the power required.

EXAMPLES.

1. Required the fourth power of $4ab^2c^3$.

Ans. $256a^4b^8c^{12}$.

2. Required the fifth power of $3ax^3y^5$.
3. Required the third power of $6xy^2z^4$.
4. Required the sixth power of $2ay^3d^2v$.
5 Required the seventh power of $2a^2bc^4$.
6. Required the sixth power of $5w^3xy^2z^4$.

(139. Let us now consider the sign with which the power should be affected.

We have seen, Art. 137, that whatever may be the sign of the monomial, its square is always positive. It is obvious from the same considerations that the product of an *even* number of negative factors is *positive ;* but the product of an *odd* number of negative factors is *negative.*

Thus

$$\begin{aligned}-a \times -a &= +a^2\\ -a \times -a \times -a &= -a^3\\ -a \times -a \times -a \times -a &= +a^4\\ -a \times -a \times -a \times -a \times -a &= -a^5\\ \text{etc.} \qquad \text{etc.} &\qquad \text{etc.}\end{aligned}$$

The product of several factors which are all *positive,* is invariably positive. Hence

Every EVEN *power is positive, but an* ODD *power has the same sign as its root.*

EXAMPLES.

1. Required the square of $-2x^5$.

Ans. $+4x^{10}$.

2. Required the square of $-3x^n$.
3. Required the cube of $-3a^3$.
4. Required the fourth power of $-3a^3b^2h$.
5. Required the fifth power of $-2a^3 \times 3x^2y$.

(140.) *A fraction is involved by involving both the numerator and denominator.*

1. Thus the square of $\frac{a}{b}$ is $\frac{a}{b} \times \frac{a}{b}$; which, by Art. 90, is equal to $\frac{a^2}{b^2}$, which by Art. 68, may be written a^2b^{-2}.

2. Required the cube of $\frac{2ab^2}{3c}$.

Ans. $\frac{8a^3b^6}{27c^3}$ or $\frac{8}{27}a^3b^6c^{-3}$.

3. Required the nth power of $\frac{a^2b}{ay^m}$.

(141.) Hence expressions with *negative* indices are involved by the same rule as those with positive indices.

Thus to find the square of a^{-3}.

This expression may be written $\frac{1}{a^3}$, which raised to the second power becomes $\frac{1}{a^6}$ or a^{-6}, the same result as would be obtained by multiplying the exponent -3 by 2.

Ex. 1. Required the square of $3a^2b^{-4}$.

Ex. 2. Required the square of $7a^{-2}b^{3}c^{-4}dx^{-1}$.

Ex. 3. Required the cube of $6ab^{-5}dy^{-2}$.

Ex. 4. Required the fourth power of $3a^{-n}b$.

Ex. 5. Required the fifth power of $2ab^{-3}c^{2}$.

(142. *A polynomial is involved by multiplying it into itself as many times less one as is denoted by the index of the power.*

Ex. 1. Required the fourth power of $a + b$.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \qquad \\ +ab+b^2 \\ \hline \end{array}$$

$(a+b)^2 = a^2 + 2ab + b^2$, the second power of $a + b$.

$$\begin{array}{r} a+b \qquad\qquad \\ \hline a^3+2a^2b+ab^2 \qquad \\ +a^2b+2ab^2+b^3 \\ \hline \end{array}$$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, the third power.

$$\begin{array}{r} a+b \qquad\qquad\qquad \\ \hline a^4+3a^3b+3a^2b^2+ab^3 \qquad \\ +a^3b+3a^2b^2+3ab^3+b^4 \\ \hline \end{array}$$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, the fourth power.

Ex. 2. Required the fourth power of $a - b$.

Ans. $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.

Ex. 3. Required the cube of $2a - 1$.

Ex. 4. Required the fourth power of $3a - h$.

Ex. 5. Required the square of $a + b + c$.

Hence it appears that *the square of a trinomial is composed of the sum of the squares of all the terms, together with twice the sum of the products of all the terms multiplied together two and two.*

Ex. 6. Required the cube of $2ab + cd$.

Ex. 7. Required the square of $a^3 + b^3$.

Ex. 8. Required the square of $a + \frac{1}{2}$.

Ex. 9. Required the cube of $x + \frac{y}{2}$.

Ex. 10. Required the square of $a + b + c + d + e$.

From this example we infer that the *square of any polynomial is composed of the sum of the squares of all the terms, together with twice the sum of the products of all the terms multiplied together, two and two*, and this proposition may be rigorously demonstrated.

It is easy to perceive that this rule for a polynomial includes the preceding rule for a trinomial, and that in Art. 60 for a binomial.

SECTION XI.

EVOLUTION AND RADICAL QUANTITIES.

(143.) *The square root of a quantity is a factor which multiplied by itself once, will produce that quantity.*

Thus the square root of a^2 is a, because a when multiplied by itself produces a^2.

The square root of 144 is 12 for the same reason.

According to Art. 22, the square root is indicated by the sign $\sqrt{\quad}$.

Thus
$$\sqrt{a^2} = a.$$
$$\sqrt{144a^2} = 12a.$$

(144.) According to Art. 136, in order to square a monomial we must square its coefficient, and multiply the exponent of each of the different letters by 2. Therefore in order to derive the square root of a monomial from its square, we must

I. *Extract the square root of its coefficient.*

II. *Divide each of the exponents by 2.*

Thus we shall have
$$\sqrt{64a^6b^4} = 8a^3b^2.$$

This is manifestly the true result, for
$$(8a^3b^2)^2 = 8a^3b^2 \times 8a^3b^2 = 64a^6b^4.$$

So also
$$\sqrt{625a^2b^8c^6} = 25ab^4c^3.$$

For $$(25ab^4c^3)^2 = 25ab^4c^3 \times 25ab^4c^3$$
$$= 625a^2b^8c^6.$$

Required the square root of $196a^2b^4c^6d^8$.

Required the square root of $225a^{2m}b^{10}x^4$.

(145.) According to Art. 140, a fraction is involved by involving both the numerator and denominator; hence it is obvious that *the square root of a fraction is equal to the root of the numerator divided by the root of the denominator.*

Thus the square root of $\frac{a^2}{b^2}$ is $\frac{a}{b}$.

1. Find the square root of $\frac{4a^2}{9x^4y^4}$.

2. Find the square root of $\frac{9a^2b^4}{16c^6d^{12}}$.

(146.) It appears from Article 144, that *a monomial cannot be the square of another monomial, unless its coefficient be a square number, and the exponents of the different letters all even numbers.*

Thus $7ab^2$ is not a perfect square, for 7 is not a *square* number, and the exponent of a is not an *even* number. Its square root may be *indicated* by the usual sign, thus $\sqrt{7ab^2}$. Expressions of this nature are called *surds*, or *radicals of the second degree.*

(147.) We have seen, Art. 137, that whatever may be the sign of a monomial, its square must be affected with the sign +. Hence we conclude that

If a monomial be positive, its square root may be either positive or negative.

Thus $$\sqrt{9a^4} = +3a^2, \text{ or } -3a^2,$$

for either of these quantities when multiplied by itself produces

$9a^4$. We therefore always affect the square root of a quantity with the double sign $\pm$, which is read *plus* or *minus*.

Thus
$$\sqrt{4a^6} = \pm 2a^3$$
$$\sqrt{25a^2b^4} = \pm 5ab^2.$$

(148.) If the monomial be affected with a *negative* sign, the extraction of its square root is *impossible;* since we have just seen that the square of every quantity whether positive or negative is necessarily positive.

Thus
$$\sqrt{-4},\ \sqrt{-5a},\ \sqrt{-9}$$
are algebraic symbols which represent operations which it is impossible to execute. Quantities of this nature are called *imaginary* or *impossible* quantities, and are symbols of absurdity which we frequently meet with in resolving quadratic equations.

Such quantities may be represented by the general form

$$\sqrt{-a}, \text{ which equals}$$
$$\sqrt{a\times-1} = \sqrt{a}\ \sqrt{-1}.$$

So that $\sqrt{a}\ \sqrt{-1}$ is a general form for all imaginary quantities of the second degree. Thus

$$\begin{aligned} \sqrt{-4} &= \sqrt{4\times-1} = 2\sqrt{-1} \\ \sqrt{-5a} &= \sqrt{5a\times-1} = \sqrt{5a}\sqrt{-1} \\ \sqrt{-9} &= \sqrt{9\times-1} = 3\sqrt{-1}. \end{aligned}$$

That is, *the square root of a negative quantity may always be represented by the square root of a positive quantity multiplied by the square root of* -1.

(149.) According to Article 138, to raise a monomial to *any* power, we raise the numerical coefficient to the given power, and multiply the exponents of each of the letters by the index of the power required. Hence reciprocally, to extract the root of any degree of a monomial, we obtain the following

RULE.

I. *Extract the root of the numerical coefficient.*

II. *Divide the exponent of each letter by the index of the required root.*

Thus $$\sqrt[3]{64a^6b^3} = 4a^3b$$
$$\sqrt[4]{16b^{12}c^{16}} = 2b^3c^4.$$

From Article 145, it is obvious that *to extract* ANY *root of a fraction, we must divide the root of the numerator by the root of the denominator.*

Thus the cube root of $\frac{27a^6b^3}{8x^3y^9}$ is $\frac{3a^2b}{2xy^3}$;

which may be written $\frac{3}{2}a^2bx^{-1}y^{-3}$.

(150.) Let us now consider the *sign* with which the root should be affected. We have seen, Article 139, that every *even* power is positive, but an *odd* power has the same sign as its root.

Thus $-a$, when raised to different powers in succession, will give

$$-a, +a^2, -a^3, +a^4, -a^5, +a^6, -a^7, \text{ etc.}$$

and $+a$ in like manner will give

$$+a, +a^2, +a^3, +a^4, +a^5, +a^6, +a^7, \text{ etc.}$$

In fact every even power (being expressed by $2n$) may be considered as the square of the nth power, or $a^{2n} = (a^n)^2$, and must therefore be positive; and in like manner every power of an uneven degree (being expressed by $2n + 1$) may be considered as the product of the $2n$th power by the original monomial, and must therefore have the same sign with the monomial.

Hence it appears

I. *That an odd root of any quantity must have the same sign as the quantity itself.*

Thus $\sqrt[3]{+8a^3} = +2a$

$$\sqrt[3]{-8a^3} = -2a$$

$$\sqrt[5]{-32a^{10}b^5} = -2a^2b$$

$$\sqrt[5]{+32a^{10}b^5} = +2a^2b$$

II. *That an even root of a positive quantity is ambiguous.*

Thus $\sqrt[4]{81a^4b^{12}} = \pm 3ab^3$

$$\sqrt[6]{64a^{18}} = \pm 2a^3.$$

III. *That an even root of a negative quantity is impossible.*

For no quantity can be found which when raised to an even power can give a negative result.

Thus $\sqrt[4]{-a}$, $\sqrt[8]{-c}$ are symbols of operations which cannot be performed, and they are therefore called *impossible* or *imaginary* quantities, as $\sqrt{-a}$, $\sqrt{-b}$ in Art. 148.

EXAMPLES.

1. Find the fourth root of $81a^8$.

Ans. $\pm 3a^2$.

2. Find the fifth root of $-243a^{10}b^5c^{-15}$.
3. Find the cube root of $-125a^3x^{-6}y^9$.
4. Find the square root of $\frac{16a^4}{9x^2y^2}$
5. Find the fifth root of $\frac{32a^{10}b^{-5}}{243}$.

(151.) According to the preceding rule, we perceive that in order that a monomial may be a perfect power of that degree whose root is required, its coefficient must be a perfect power of that degree, and the exponent of each letter must be divisible by the index of the root.

When the monomial whose root is required is *not* a perfect power of the given degree, we can only *indicate* the operation to be performed. Thus if it be required to extract the cube root of $4a^2b^5$, the operation may be indicated by writing the expression thus

$$\sqrt[3]{4a^2b^5}.$$

Expressions of this nature are called *surds*, or *irrational quantities*, or *radicals of the second, third* or *n*th *degree*, according to the index of the root required.

(152.) The method of extracting the roots of polynomials will be considered in Section XVII. There is, however, one class so simple and of so frequent occurrence that it may properly be introduced here. In Arts. 60 and 61 we have seen that

the square of $a + b$ is $a^2 + 2ab + b^2$,
and the square of $a - b$ is $a^2 - 2ab + b^2$.

Hence the square root of $a^2 \pm 2ab + b^2$ is $a \pm b$.

Hence *a trinomial is a perfect square, when two of its terms are squares, and the third is the double product of the roots of these squares.*

Whenever, therefore, we meet with a quantity of this description, we may know that its square root is a binomial, and this may be found *by extracting the roots of the two terms which are complete squares, and connecting them by the sign of the other term.*

Ex. 1. To find the square root of $a^2 + 4ab + 4b^2$.

The two terms, a^2 and $4b^2$ are complete squares, and the third term $4ab$ is twice the product of the roots a and $2b$; hence $a + 2b$ is the root required.

Ex. 2. Find the square root of $9a^2 - 24ab + 16b^2$.

Ex. 3. Find the square root of $9a^4 - 30a^3b + 25a^2b^2$.

(153.) *No binomial can be a perfect square.* For the square of a monomial is a monomial; and the square of a binomial consists of three distinct terms which do not admit of being reduced with each other.

Thus such an expression as

$$a^2 + b^2$$

is not a square; it wants the term $\pm 2ab$ to render it the square of $a \pm b$. This remark should be continually borne in mind, as beginners are very prone to put the square root of $a^2 + b^2$ equal to $a + b$.

IRRATIONAL QUANTITIES, OR SURDS.

(154.) *Irrational quantities, or surds, are quantities affected with a radical sign, and which have no exact root, or a root which can be exactly expressed in numbers.*

Thus $\sqrt{3}$ is a surd, because the square root of 3 cannot be expressed in numbers with perfect exactness.

In decimals it is 1.7320508 nearly.

A *rational quantity* is one which can be expressed in finite terms, or without any radical sign or fractional index; as a, $5a^2$, etc.

(155.) We have seen, Art. 144, that in order to extract the square root of a monomial, we must divide each of its exponents by 2.

Thus the square root of a^2 is a^1 or a; that of a^4 is a^2; that of a^6 is a^3 and so on; and as this principle is general, the square root of a^3 must necessarily be $a^{\frac{3}{2}}$, and that of a^5 must be $a^{\frac{5}{2}}$; consequently we shall in the same manner have $a^{\frac{1}{2}}$ for the square root of a^1. Whence we see that

$$a^{\frac{1}{2}} \text{ is equal to } \sqrt{a}$$

$a^{\frac{3}{2}}$ is the same as $\sqrt[2]{a^3}$

$a^{\frac{5}{2}}$ is equivalent to $\sqrt{a^5}$

etc. etc.

We have also seen, Art. 149, that in order to extract the root of any degree of a monomial, we must divide the exponent of each letter by the index of the required root.

Thus the cube root of a^3 is a^1, or a; the cube root of a^6 is a^2; the cube root of a^9 is a^3, and so on. So also the cube root of a^2 is $a^{\frac{2}{3}}$; the cube root of a^4 is $a^{\frac{4}{3}}$; the cube root of a, or a^1 is $a^{\frac{1}{3}}$. Whence it appears that

$a^{\frac{1}{3}}$ is the same as $\sqrt[3]{a}$

$a^{\frac{4}{3}}$ is equivalent to $\sqrt[3]{a^4}$

$a^{\frac{5}{3}}$ is equivalent to $\sqrt[3]{a^5}$

etc. etc.

In the same manner the fourth root of a is $a^{\frac{1}{4}}$, which expression has therefore the same value as $\sqrt[4]{a}$; the fifth root of a will be $a^{\frac{1}{5}}$, which is conseqently equivalent to $\sqrt[5]{a}$, and the same observation may be extended to all roots of a higher degree.

(156.) Other fractional exponents are to be understood in the same way. Thus if we have $a^{\frac{4}{3}}$, this means that we must first take the fourth power of a, and then extract its cube or third root; so that $a^{\frac{4}{3}}$ is the same as $\sqrt[3]{a^4}$.

So also to find the value of $a^{\frac{3}{4}}$, we must first take the cube or the third power of a, which is a^3, and then extract the fourth root of that power; so that $a^{\frac{3}{4}}$ is the same as $\sqrt[4]{a^3}$,

and $a^{\frac{4}{5}}$ is equal to $\sqrt[5]{a^4}$.

Hence, *the numerator of a fractional index denotes the power, and the denominator the root to be extracted.*

Again, let it be required to extract the cube root of $\frac{1}{a^4}$.

In the first place $\frac{1}{a^4} = a^{-4}$. Now to extract the cube root of a^{-4} we must divide its exponent by 3, which gives us

$$a^{-\frac{4}{3}}.$$

But the cube root of $\frac{1}{a^4}$ may also be represented by

$$\frac{1}{a^{\frac{4}{3}}}.$$

Hence $\frac{1}{a^{\frac{4}{3}}}$ is equivalent to $a^{-\frac{4}{3}}$

So also $\frac{1}{a^{\frac{1}{2}}}$ is equivalent to $a^{-\frac{1}{2}}$

$\frac{1}{a^{\frac{1}{3}}}$ " $a^{-\frac{1}{3}}$

$\frac{1}{a^{\frac{1}{n}}}$ " $a^{-\frac{1}{n}}$.

Thus we see that the principle of Art. 69, that a factor may be transferred from the numerator to the denominator of a fraction, or from the denominator to the numerator by changing the sign of its index, is applicable also to *fractional indices.*

We may, therefore, entirely reject the radical signs hitherto made use of, and employ in their stead the fractional exponents which we have just explained; and indeed many of the difficulties in the calculation of radical quantities disappear when fractional exponents are substituted for the radical signs.

CASE I.

To reduce surds to their most simple forms.

157.) Surds may frequently be simplified by the application of the following principle; *the square root of the product of two or more factors is equal to the product of the square roots of those factors.*

Or in algebraic language

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}.$$

For each member of this equation squared, will give the same quantity.

Thus the square of $\sqrt{ab}$ is ab.

And the square of $\sqrt{a} \times \sqrt{b}$ is $(\sqrt{a})^2 \times (\sqrt{b})^2 = ab$.

Hence, since the *squares* of the quantities $\sqrt{ab}$ and $\sqrt{a} \times \sqrt{b}$ are equal, the quantities themselves must be equal.

Take the example $\sqrt{4a}$.

This may be put under the form $\sqrt{4} \times \sqrt{a}$.

But $\sqrt{4}$ is equal to 2.

Hence, $\sqrt{4a} = \sqrt{4} \times \sqrt{a} = 2\sqrt{a} = 2a^{\frac{1}{2}}$

$2\sqrt{a}$ is considered a *simpler* form than $\sqrt{4a}$ for reasons which will be better understood hereafter.

Again, reduce $\sqrt{48}$ to its most simple form.

$$\sqrt{48} \text{ is equal to } \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4\sqrt{3}.$$

In general, therefore, in order to simplify a monomial radical of the second degree, *separate it into two factors, one of which is a perfect square, extract its root, and prefix it to the other factor with the radical sign between them.*

In the expressions $2\sqrt{a}$ and $4\sqrt{3}$, the quantities 2 and 4 are called *the coefficients of the radical.*

EXAMPLES.

1. Reduce $2\sqrt{32}$ to its most simple form.

Ans. $8\sqrt{2}$.

2. Reduce $\sqrt{125a^3}$ to its most simple form.
3. Reduce $\sqrt{98ab^4}$ to its most simple form.

4. Reduce $\sqrt{294ab^2}$ to its most simple form.
5. Reduce $7\sqrt{80abc^3}$ to its most simple form.
6. Reduce $\sqrt{98a^2x^6y^2}$ to its most simple form.
7. Reduce $\sqrt{45a^2b^3c^2d}$ to its most simple form.
8. Reduce $\sqrt{864a^2b^5c^{11}}$ to its most simple form.

(158.) Surds of *any* degree may be simplified by the application of the following principle, which is merely an extension of that already proved in the preceding Article.

The n*th root of the product of any number of factors is equal to the product of the* n*th roots of those factors.*

Or in algebraic language

$$\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}.$$

For raise each of these expressions to the nth power and we obtain the same result.

Thus the nth power of $\sqrt[n]{ab}$ is ab.

And the nth power of $\sqrt[n]{a} \times \sqrt[n]{b}$ is $(\sqrt[n]{a})^n \times (\sqrt[n]{b})^n = ab$.

Hence, since the *same* powers of the quantities $\sqrt[n]{ab}$ and $\sqrt[n]{a} \times \sqrt[n]{b}$ are equal, the quantities themselves must be equal.

Thus take the expression

$$\sqrt[3]{8a^2}.$$

This is equivalent to $\sqrt[3]{8} \times \sqrt[3]{a^2}$, which is equal to $2\sqrt[3]{a^2}$.

Again, take the expression

$$\sqrt[4]{48a^5}.$$

This is equivalent to $\sqrt[4]{16a^4} \times \sqrt[4]{3a}$, which is equal to $2a\sqrt[4]{3a}$.

Hence, to simplify a monomial radical of any degree, we have the following

RULE.

Separate the quantity into two factors, one of which is an exact power of the same name with the root; extract its root; and prefix it to the other factor with the radical sign between them.

In the above expressions, the quantities 2 and $2a$ placed before the radical sign, are called the *coefficients of the radical.*

EXAMPLES.

1. Reduce $\sqrt[3]{56a^5b^6}$ to its most simple form.

Ans. $2ab^2\sqrt[3]{7a^2}$.

2. Reduce $\sqrt[3]{54a^4b^3c^2}$ to its most simple form.
3. Reduce $\sqrt[4]{48a^5b^8c^6}$ to its most simple form.
4. Reduce $\sqrt[6]{192a^7bc^{12}}$ to its most simple form.
5. Reduce $\sqrt[3]{192a^4b^2c^3}$ to its most simple form.
6. Reduce $9\sqrt[3]{81b^2}$ to its most simple form.

(159.) There is another principle which can frequently be employed to advantage in simplifying radicals.

The *square* of the *cube* of a is equal to the *sixth power* of a.

For the square of the cube of a is $a^3 \times a^3$, which equals $a^{3+3} = a^6$.

So also the *fourth power* of the *cube* of a, is equal to the *twelfth power* of a.

For
$$\begin{aligned} (a^3)^4 &= a^3 \times a^3 \times a^3 \times a^3 \\ &= a^{3+3+3+3} \\ &= a^{12}. \end{aligned}$$

And in general, the mth power of the nth power of any quantity, is equal to the mnth power of that quantity.

That is $(a^n)^m = a^{mn}$.

Hence reciprocally,

The mnth root of any quantity is equal to the mth root of the nth root of that quantity.

Thus, the fourth root = the square root of the square root,
The sixth root = the square root of the cube root,
or the cube root of the square root.

The eighth root = the square root of the fourth root,
or the fourth root of the square root,
The ninth root = the cube root of the cube root.

Hence, *when the index of the root is the product of two or more factors, we may obtain the root required by extracting in succession the roots denoted by those factors.*

Ex. 1. Let it be required to extract the sixth root of 64.

The sixth root is equal to the cube root of the square root.

The square root of 64 is 8,
and the cube root of 8 is 2.

Hence the sixth root of 64 is 2.

Ex. 2. Let it be required to extract the eighth root of 256.

The eighth root is equal to the fourth root of the square root; or to the square root of the square root, of the square root.

The square root of 256 is 16,
and the fourth root of 16 is 2.

Hence, the eighth root of 256 is 2.

When one of the roots can be extracted, and the other not, a radical may be *simplified* by extracting one of the roots.

Thus, the fourth root of 9 is equal to the square root of the square root of 9; that is the square root of 3.

Or algebraically $\sqrt[4]{9} = \sqrt{3}$.

Ex. 3. Reduce $\sqrt[6]{4a^2}$ to its most simple form.

Ex. 4. Reduce $\sqrt[4]{36a^2b^2}$ to its most simple form.

Ex. 5. Reduce $\sqrt[mn]{a^n}$ to its most simple form.

Ex. 6. Reduce $\sqrt[8]{25a^4b^2c^6}$ to its most simple form.

CASE II.

(160.) *To reduce a rational quantity to the form of a surd.*

The square root of the square of a is obviously a; that is

$$a = \sqrt{a^2} = a^{\frac{2}{2}}.$$

So also the cube root of the cube of a is a;

that is $$a = \sqrt[3]{a^3} = a^{\frac{3}{3}}.$$

Hence, to reduce a rational quantity to the form of a surd, we have the following

RULE.

Raise the quantity to a power of the same name with the given surd, and then apply the corresponding radical sign or index.

EXAMPLES.

1. Reduce 3 to the form of the square root.

   Here $3 \times 3 = 3^2 = 9$; whence $3 = \sqrt{9}$. Ans.

2. Reduce ax to the form of the square root.

   Ans. $\sqrt{a^2x^2}$ or $(a^2x^2)^{\frac{1}{2}}$.

3. Reduce $2x^2$ to the form of the cube root.
4. Reduce 5 to the form of the square root.
5. Reduce $-3x$ to the form of the cube root.
6. Reduce $-\frac{1}{2}x^2$ to the form of the fourth root.
7. Reduce a^3b^4 to the form of the square root.

It will be observed that this Case is nearly the reverse of the preceding, and consequently brings quantities into a *less simple* form; nevertheless this form is sometimes better suited to subsequent operations, as will be seen hereafter.

CASE III.

(161.) *To reduce quantities which have different indices to others of the same value having a common index.*

Example 1. To reduce $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$ to a common index.

From the preceding Article it is obvious that the square ro of a is equal to the sixth root of the cube of a;

that is $$a^{\frac{1}{2}} = a^{\frac{3}{6}} = \sqrt[6]{a^3}.$$

So also $$a^{\frac{1}{3}} = a^{\frac{2}{6}} = \sqrt[6]{a^2}.$$

Thus the quantities $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$ are reduced to $a^{\frac{3}{6}}$ and $a^{\frac{2}{6}}$ which are of the same value, and have the common index $\frac{1}{6}$.

Ex. 2. Reduce $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ to a common index.

$$3^{\frac{1}{2}} = 3^{\frac{3}{6}} = (3^3)^{\frac{1}{6}} = 27^{\frac{1}{6}}$$

$$2^{\frac{1}{3}} = 2^{\frac{2}{6}} = (2^2)^{\frac{1}{6}} = 4^{\frac{1}{6}}.$$

Hence $\sqrt[6]{27}$ and $\sqrt[6]{4}$ are the quantities required.

Hence we derive the following

RULE.

Reduce the indices to a common denominator; raise each quantity to the power denoted by the numerator of its reduced index; and take the root denoted by the common denominator.

Ex. 3. Reduce $2^{\frac{1}{3}}$ and $4^{\frac{1}{4}}$ to a common index.

Ex. 4. Reduce a^2 and $a^{\frac{1}{2}}$ to a common index.

Ex. 5. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{2}{3}}$ to a common index.

Ex. 6. Reduce $5^{\frac{2}{3}}$ and $7^{\frac{3}{4}}$ to a common index.

Ex. 7. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to a common index.

(162.) If it is required to reduce a quantity to a *given* index, *Divide the index of the quantity by the given index, place the quotient over the quantity, and set the given index over the whole.*

Reduce $a^{\frac{1}{6}}$ to the index $\frac{1}{3}$.

$$\tfrac{1}{6} \div \tfrac{1}{3} = \tfrac{1}{2}.$$

Hence $a^{\frac{1}{6}} = (a^{\frac{1}{2}})^{\frac{1}{3}} = \sqrt[3]{a^{\frac{1}{2}}}$. Ans.

Reduce $3^{\frac{1}{2}}$ and $4^{\frac{1}{4}}$ to quantities that shall have the common index $\frac{1}{8}$.

CASE IV.

To add surd quantities together.

(163.) Two radicals are *similar*, when they have the same index, and the same quantity under the sign.

Thus, $3\sqrt{a}$ and $5\sqrt{a}$ are similar radicals.

So also $7\sqrt[3]{b}$ and $10\sqrt[3]{b}$ are similar radicals.

But $\sqrt{a}$ and $\sqrt[3]{a}$ are *not* similar radicals, for although they have the same quantity under the radical sign, they have not the same index.

Ex. 1. Find the sum of $2\sqrt{a}$ and $3\sqrt{a}$.

As these are *similar* radicals, we may unite their coefficients by the usual rule, for it is evident that twice the square root of a and three times the square root of a, make five times the square root of a. Hence the following

RULE.

When the radicals are similar, add the coefficients, and annex the radical part.

But if the quantities are dissimilar, and cannot be made similar by the reductions in the preceding articles, they can only be connected together by the sign of addition.

Ex. 2. Add $\sqrt{6}$ to $2\sqrt{6}$.

Ans. $3\sqrt{6}$.

Ex. 3. Add $5\sqrt[3]{a}$ and $-2\sqrt[3]{a}$.

Ex. 4. Add $a\sqrt{b+c}$ and $x\sqrt{b+c}$.

If the radical parts are *originally different*, they must if possible be made alike by the preceding methods.

Ex. 5. Add $\sqrt{27}$ to $\sqrt{48}$.

Here $\sqrt{27}=\sqrt{9\times 3}=3\sqrt{3}$.

And $\sqrt{48}=\sqrt{16\times 3}=4\sqrt{3}$.

Whence $7\sqrt{3}=$ the sum.

Ex. 6. Add $\sqrt[3]{500}$ and $\sqrt[3]{108}$.

Ans. $8\sqrt[3]{4}$.

Ex. 7. Add $4\sqrt{147}$ and $3\sqrt{75}$.

Ans. $43\sqrt{3}$.

Ex. 8. Add $3\sqrt{\tfrac{2}{5}}$ and $2\sqrt{\tfrac{1}{10}}$.

Here $3\sqrt{\tfrac{2}{5}} = 3\sqrt{\tfrac{10}{25}} = \tfrac{3}{5}\sqrt{10}$.

And $2\sqrt{\tfrac{1}{10}} = 2\sqrt{\tfrac{10}{100}} = \tfrac{2}{10}\sqrt{10}$.

Whence $\tfrac{4}{5}\sqrt{10}$ = the sum.

Ex. 9. Add together $\sqrt{72}$ and $\sqrt{128}$.

Ex. 10. Add together $\sqrt{180}$ and $\sqrt{405}$.

Ex. 11. Add together $\sqrt[3]{40}$ and $\sqrt[3]{135}$.

Ex. 12. Add together $8\sqrt[4]{32}$ and $5\sqrt[4]{2}$.

CASE V.

To find the difference of surd quantities.

(164.) It is evident that the subtraction of surd quantities may be performed in the same manner as addition, except that the signs in the subtrahend are to be changed according to Article 45.

Ex. 1. Required to find the difference of $\sqrt{448}$ and $\sqrt{112}$.

Here $\sqrt{448} = \sqrt{64 \times 7} = 8\sqrt{7}$.

And $\sqrt{112} = \sqrt{16 \times 7} = 4\sqrt{7}$.

Whence $4\sqrt{7}$ = the difference.

Ex. 2. Find the difference between $\sqrt[3]{192}$ and $\sqrt[3]{24}$.

Here $\sqrt[3]{192} = \sqrt[3]{64 \times 3} = 4\sqrt[3]{3}$.

And $\sqrt[3]{24} = \sqrt[3]{8 \times 3} = 2\sqrt[3]{3}$.

Whence $2\sqrt[3]{3}$ = the difference.

Ex. 3. Find the difference between $5\sqrt{20}$ and $3\sqrt{45}$.

Here $5\sqrt{20} = 5\sqrt{4 \times 5} = 10\sqrt{5}$.

And $3\sqrt{45} = 3\sqrt{9 \times 5} = 9\sqrt{5}$.

Whence $\sqrt{5}$ = the difference.

Ex. 4. Find the difference of $2\sqrt{50}$ and $\sqrt{18}$.

Ex. 5. Find the difference of $2\sqrt{320}$ and $3\sqrt{40}$.

Ex. 6. Find the difference of $\sqrt{80a^4x}$ and $\sqrt{20a^2x^3}$.

Ex. 7. Find the difference of $\sqrt{162a^2}$ and $2\sqrt{72a^2}$.

CASE VI.

To multiply surd quantities together.

(165.) Let it be required to multiply $\sqrt[n]{a}$ by $\sqrt[n]{b}$.

The result will be $\sqrt[n]{ab}$.

For raise each of these quantities to the power of n, and we obtain the same result ab; hence these two expressions are equal. We thus obtain the following

RULE.

When the surds have the same index, multiply the quantities under the sign by each other, and affect the result with the common radical sign. If there be coefficients, these must be multiplied separately.

Ex. 1. Required the product of $3\sqrt{8}$ and $2\sqrt{6}$.

Here $3\sqrt{8}$.

Multiplied by $2\sqrt{6}$.

Gives $6\sqrt{48} = 6\sqrt{16 \times 3} = 24\sqrt{3}$. Ans.

Proof. Square $3\sqrt{8}$, and we obtain $9 \times 8 = 72$.

Square $2\sqrt{6}$, and we obtain $4 \times 6 = 24$.

72 multiplied by 24 = 1728.

Also $24\sqrt{3}$ squared $= 576 \times 3 = 1728$.

Ex. 2. Required the product of $5\sqrt{8}$ and $3\sqrt{5}$.

Ex. 3. Required the product of $7\sqrt[3]{18}$ and $5\sqrt[3]{4}$.

Ex. 4. Required the product of $\frac{1}{4}\sqrt[3]{6}$ and $\frac{3}{15}\sqrt[3]{17}$.

Ex. 5. Required the product of $\frac{1}{2}\sqrt[3]{18}$ and $5\sqrt[3]{20}$.

If the surds have *not* the same index, they must first be reduced to a common index.

Ex. 6. Required the product of $\sqrt{2}$ and $\sqrt[3]{3}$.

Here $\sqrt{2} = 2^{\frac{3}{6}} = (2^3)^{\frac{1}{6}} = \sqrt[6]{8}$.

And $\sqrt[3]{3} = 3^{\frac{2}{6}} = (3^2)^{\frac{1}{6}} = \sqrt[6]{9}$.

Whence $\sqrt[6]{72}$ = the product.

(166.) We have seen in Article 50, that *powers* of the same quantity may be multiplied by adding their exponents. The same principle may be extended to *roots* of the same quantity.

Let it be required to multiply $\sqrt{a}$ by $\sqrt[3]{a}$, or $a^{\frac{1}{2}}$ by $a^{\frac{1}{3}}$.

We have seen in Art. 155, that $a^{\frac{1}{2}} = a^{\frac{3}{6}}$, and $a^{\frac{1}{3}} = a^{\frac{2}{6}}$.

But $a^{\frac{3}{6}} = a^{\frac{1}{6}} \times a^{\frac{1}{6}} \times a^{\frac{1}{6}}$.

And $a^{\frac{2}{6}} = a^{\frac{1}{6}} \times a^{\frac{1}{6}}$.

The product therefore is $a^{\frac{1}{6}} \times a^{\frac{1}{6}} \times a^{\frac{1}{6}} \times a^{\frac{1}{6}} \times a^{\frac{1}{6}} = a^{\frac{5}{6}}$.

Hence, *roots of the same quantity may be multiplied by adding their fractional exponents.*

Ex. 1. Multiply $5a^{\frac{1}{2}}$ by $3a^{\frac{1}{3}}$.

Ans. $15a^{\frac{5}{6}}$.

Ex. 2. Multiply $3a^{\frac{2}{3}}$ by $21a^{\frac{1}{4}}$.

Ex. 3. Multiply $3x^{\frac{1}{2}}y^{\frac{1}{3}}$ by $4x^{\frac{1}{2}}y^{\frac{2}{3}}$.

Ex. 4. Multiply $(a+b)^{\frac{1}{n}}$ by $(a+b)^{\frac{1}{m}}$.

(167.) If the rational quantities, instead of being *coefficients* of the radical quantities, are connected with them by the signs + or —, each term in the multiplier must be multiplied into each term of the multiplicand.

1. Let it be required to multiply $3+\sqrt{5}$

$$\begin{array}{lrl} & & 3+\sqrt{5} \\ & & \text{by } 2-\sqrt{5} \\ \hline & & 6+2\sqrt{5} \\ & & \quad -3\sqrt{5}-5 \\ \hline \text{We obtain the product} & & 6-\sqrt{5}-5 \\ \text{which reduces to} & & 1-\sqrt{5}. \end{array}$$

2. Multiply $7+2\sqrt{6}$ by $9-5\sqrt{6}$.

Ans. $3-17\sqrt{6}$.

3. Multiply $9+2\sqrt{10}$ by $9-2\sqrt{10}$.

Ans. 41.

CASE VII.

To divide one surd quantity by another.

(168.) Let it be required to divide $\sqrt[6]{a^3}$ by $\sqrt[6]{a^2}$.

The quotient must be a quantity which multiplied by the divisor shall produce the dividend; we thus obtain $\sqrt[6]{a}$, for according to Art. 165, $\sqrt[6]{a^2}\times\sqrt[6]{a}=\sqrt[6]{a^3}$.

That is $$\frac{\sqrt[6]{a^3}}{\sqrt[6]{a^2}}=\sqrt[6]{a}.$$

Hence the following

RULE.

Quantities under the same radical sign or index may be divided like rational quantities, the quotient being affected with the common radical sign. If there are coefficients, they must be divided separately.

If the radicals have *not* the same index, we must first reduce them to a common index.

EXAMPLES.

1. It is required to divide $8\sqrt{108}$ by $2\sqrt{6}$.

Here $\dfrac{8\sqrt{108}}{2\sqrt{6}} = 4\sqrt{18} = 4\sqrt{9 \times 2} = 12\sqrt{2}$. Ans.

2. Divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.

Here $\dfrac{8\sqrt[3]{512}}{4\sqrt[3]{2}} = 2\sqrt[3]{256} = 2\sqrt[3]{64 \times 4} = 8\sqrt[3]{4}$. Ans.

3. Divide $6\sqrt[3]{54}$ by $3\sqrt[3]{2}$.
4. Divide $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$.
5. Divide $4\sqrt{6a^2y}$ by $2\sqrt{3y}$.
6. Divide $16(a^3b)^{\frac{1}{m}}$ by $8(ac)^{\frac{1}{m}}$.
7. Divide $4\sqrt[3]{12}$ by $2\sqrt{3}$.

As the radicals in this last example have not the same index, they must be reduced to a common index.

$$4\sqrt[3]{12} = 4(12)^{\frac{1}{3}} = 4(12)^{\frac{2}{6}} = 4(144)^{\frac{1}{6}}.$$

$$2\sqrt{3} = 2(3)^{\frac{1}{2}} = 2(3)^{\frac{3}{6}} = 2(27)^{\frac{1}{6}}.$$

$$\frac{4(144)^{\frac{1}{6}}}{2(27)^{\frac{1}{6}}} = 2\left(\tfrac{144}{27}\right)^{\frac{1}{6}} = 2\left(\tfrac{16}{3}\right)^{\frac{1}{6}} = 2\sqrt[6]{\tfrac{16}{3}}.$$

(169.) We have seen in Art. 67, that in order to divide quantities expressed by the same letter, we must subtract the exponent of the divisor from the exponent of the dividend. The same principle may be extended to fractional exponents.

Thus let it be required to divide $a^{\frac{1}{2}}$ by $a^{\frac{1}{3}}$.
Then by the preceding article

$$\frac{a^{\frac{1}{2}}}{a^{\frac{1}{3}}} = \frac{a^{\frac{3}{6}}}{a^{\frac{2}{6}}} = \sqrt[6]{\frac{a^3}{a^2}} = \sqrt[6]{a} = a^{\frac{1}{6}}.$$

Hence, *a root is divided by another root of the same letter or quantity, by subtracting the exponent of the divisor from that of the dividend.*

Ex. 1. Divide $(ab)^{\frac{2}{3}}$ by $(ab)^{\frac{1}{3}}$.

Ans.

Ex. 2. Divide a^3 by $a^{\frac{1}{2}}$.

Ex. 3. Divide $a^{\frac{2}{3}}$ by $a^{\frac{1}{4}}$.

Ex. 4. Divide $a^{\frac{1}{n}}$ by $a^{\frac{1}{m}}$.

Ex. 5. Divide $4\sqrt{ab}$ by $2\sqrt[3]{ab}$.

CASE VIII.

(170.) *To raise surd quantities to any power.*

Let it be required to find the square of $a^{\frac{1}{3}}$.

The square of a quantity is found by multiplying it into itself once.

Hence, the square of $a^{\frac{1}{3}}$ is equal to $a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}} = a^{\frac{2}{3}}$.

That is $\qquad (a^{\frac{1}{3}})^2 = a^{\frac{2}{3}}$.

Again, let it be required to find the cube of $a^{\frac{1}{5}}$.

The cube of a quantity is found by multiplying it into itself twice.

Hence the cube of $a^{\frac{1}{5}}$ is equal to $a^{\frac{1}{5}} \times a^{\frac{1}{5}} \times a^{\frac{1}{5}} = a^{\frac{3}{5}}$.

That is $\qquad (a^{\frac{3}{5}})^3 = a^{\frac{3}{5}}$.

In the same manner we should find the nth power of $a^{\frac{1}{m}} = a^{\frac{n}{m}}$.

Hence we have the following

RULE.

Radical quantities are involved by multiplying their fractional indices by the index of the required power.

Ex. 1. Required the fourth power of $a^{\frac{1}{3}}$.

Ex. 2. Required the fourth power of $\frac{2}{3}a^{\frac{1}{3}}$.

Ex. 3. Required the cube of $\frac{2}{3}\sqrt{3}$.

Ans. $\frac{8}{9}\sqrt{3}$.

Ex. 4. Required the square of $3\sqrt[3]{3}$.

Ex. 5. Required the fourth power of $\frac{1}{6}\sqrt{6}$.

(171.) *If the radical quantities are connected with others by the signs + and —, they must be involved by a multiplication of the several terms.*

Ex. 1. Required the square of $3+\sqrt{5}$

$$\begin{array}{lr}
 & 3+\sqrt{5} \\
 & 3+\sqrt{5} \\
\hline
 & 9+3\sqrt{5} \qquad \\
 & 3\sqrt{5}+5 \\
\hline
\text{The square is} & 9+6\sqrt{5}+5
\end{array}$$

or $14+6\sqrt{5}$. Ans.

Ex. 2. Required the square of $3+2\sqrt{5}$.

These two examples are comprehended under the rule in Art. 60, that the square of the sum of two quantities, is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.

Ex. 3. Reqired the cube of $\sqrt{x}+3\sqrt{y}$.

Ex. 4. Required the fourth power of $\sqrt{3}-\sqrt{2}$.

CASE IX.

To find the roots of surd quantities.

(172.) A root of a quantity is that which multiplied by itself a certain number of times will produce the given quantity. But we have seen that a radical quantity is involved by multiplying its index into the index of the required power. Hence

To find the roots of surd quantities,

Divide the fractional index by the number expressing the root to be found.

Thus, the square root of $a^{\frac{1}{3}}$ is $a^{\frac{1}{3} \div 2} = a^{\frac{1}{6}}$.

For by Art. 170, we obtain the square of $a^{\frac{1}{6}}$ by multiplying the index $\frac{1}{6}$ by 2.

That is $$(a^{\frac{1}{6}})^2 = a^{\frac{2}{6}} = a^{\frac{1}{3}}.$$

EXAMPLES.

1. Find the square root of $9\,(3)^{\frac{1}{3}}$.

Here $(9(3)^{\frac{1}{3}})^{\frac{1}{2}} = 9^{\frac{1}{2}} \times 3^{\frac{1}{3} \div 2} = 3\,(3)^{\frac{1}{6}} = 3\,\sqrt[3]{3}$. Ans.

2. Required the cube root of $\frac{1}{8}\sqrt{2}$.

Ans. $\frac{1}{2}\sqrt[6]{2}$.

3. Required the square root of 10^3.
4. Required the cube root of $\frac{8}{27}a^4$.
5. Required the fourth root of $\frac{16}{81}a^{\frac{2}{3}}$.
6. Required the cube root of $\frac{64}{125}a^6$.
7. Required the cube root of $\frac{a}{3}\sqrt{\frac{a}{3}}$.

CASE X.

To find multipliers which shall cause surds to become rational.

(173.) I. When the surd is a *monomial.*

The quantity $\sqrt{a}$ is rendered rational by multiplying by $\sqrt{a}$.

For $$\sqrt{a} \times \sqrt{a} = a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a.$$

So also $a^{\frac{1}{3}}$ is rendered rational by multiplying by $a^{\frac{2}{3}}$.

For $$a^{\frac{1}{3}} \times a^{\frac{2}{3}} = a^{\frac{3}{3}} = a.$$

Also $a^{\frac{1}{4}}$ is rendered rational by multiplying by $a^{\frac{3}{4}}$.

For $$a^{\frac{1}{4}} \times a^{\frac{3}{4}} = a^{\frac{4}{4}} = a.$$

In general $a^{\frac{1}{n}}$ is rendered rational by multiplying by $a^{\frac{n-1}{n}}$.

For $$a^{\frac{1}{n}} \times a^{\frac{n-1}{n}} = a^{\frac{n-1+1}{n}} = a^{\frac{n}{n}} = a.$$

Hence we deduce the following

RULE.

Multiply the surd by the same quantity having such an exponent, as when added to the exponent of the given surd shall be equal to unity.

II. When the surd is a *binomial.*

If the binomial contains only the *square root, multiply the given binomial by the same expression with the sign of one of its terms changed, and it will give a rational product.*

Ex. 1. Thus $\sqrt{a} + \sqrt{b}$

Multiplied by $\sqrt{a} - \sqrt{b}$

$$\begin{array}{l} a + \sqrt{ab} \\ \quad\; - \sqrt{ab} - b \end{array}$$

Gives a product $a \qquad - b$, which is rational.

Ex. 2. Find a multiplier which shall render $5 + \sqrt{3}$ rational.

Given surd $5 + \sqrt{3}$

Multiplier $5 - \sqrt{3}$

Product $25 - 3 = 22$ as required.

These two examples are comprehended under the Rule in Art. 62, the product of the sum and difference of two quantities is equal to the difference of their squares.

Ex. 3. Find a multiplier that shall make $\sqrt{5}+\sqrt{3}$ rational.

Ex. 4. Find a multiplier that shall make $\sqrt{5}-\sqrt{x}$ rational.

Ex. 5. Find a multiplier that shall make $\sqrt{a}-\sqrt{abc}$ rational.

III. When the surd is a *trinomial*, it may be reduced first to a binomial surd, and then to a rational quantity.

Ex. 1. To find multipliers that shall make $\sqrt{5}+\sqrt{3}-\sqrt{2}$ rational.

$$\begin{array}{ll}
\text{Given surd} & \sqrt{5}+\sqrt{3}-\sqrt{2} \\
\text{First multiplier} & \sqrt{5}+\sqrt{3}+\sqrt{2} \\
\hline
 & 5+\sqrt{15}-\sqrt{10} \\
 & \quad +\sqrt{15}+3 \qquad -\sqrt{6} \\
 & \qquad\qquad +\sqrt{10}+\sqrt{6}-2 \\
\hline
\text{First product} & 6+2\sqrt{15} \\
\text{Second multiplier} & -6+2\sqrt{15} \\
\hline
 & -36-12\sqrt{15} \\
 & \qquad +12\sqrt{15}+60 \\
\hline
\text{Second product} & -36+60=24 \text{ a rational quantity.}
\end{array}$$

Ex. 2. To find multipliers that shall make $\sqrt{a}+\sqrt{b}+\sqrt{c}$ rational.

CASE XI.

(174.) *To reduce a fraction containing surds, to another having a rational numerator or denominator.*

RULE.

Multiply both numerator and denominator by a factor which will render either of them rational, as the case may require.

Ex. 1. If both terms of the fraction $\frac{\surd a}{\surd b}$ be multiplied by $\surd a$, it will become $\frac{a}{\surd ab}$ in which the *numerator* is rational.

Or if both terms be multiplied by $\surd b$, it will become $\frac{\surd ab}{b}$, in which the *denominator* is rational.

Ex. 2. Reduce the fraction $\frac{2}{\surd 3}$ to one that shall have a rational denominator.

Ans. $\frac{2\surd 3}{3}$.

Ex. 3. Reduce $\frac{3}{\surd 5 - \surd 2}$ to a fraction having a rational denominator.

Ans. $\surd 5 + \surd 2$.

Ex. 4. Reduce $\frac{\surd 2}{3 - \surd 2}$ to a fraction having a rational denominator.

Ans. $\frac{2}{7} + \frac{3}{7} \surd 2$.

Ex. 5. Reduce $\frac{a - \surd b}{a + \surd b}$ to a fraction having a rational denominator.

Ex. 6. Reduce $\frac{8}{\sqrt{3}+\sqrt{2}+1}$ to a fraction having a rational denominator.

Ex. 7. Reduce $\frac{\sqrt{5}+\sqrt{2}}{3}$ to a fraction having a rational numerator.

(175.) The *utility* of the preceding transformations may be illustrated by computing the *numerical value* of a fractional surd.

Suppose it is required to find the value of the fraction $\frac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}}$; and to make the illustration more striking, suppose *great accuracy* is required, as for example that the result shall be correct to 20 decimal places. We might first extract the square root of 5 to 20 decimal places, then multiply this root by 7. We thus get the value of the given numerator. Then extract in succession the square roots of 11 and 3 to 20 decimal places, take their sum, and divide by it the value of the numerator just found; it will give the value of the fraction. Observe that we have extracted the square root of *three* different numbers to 20 places, and have divided 22 figures by 21 figures, and after all we cannot be sure that our result is correct to more than 19 decimal places.

Suppose now we commence with making the denominator of the given fraction rational. The fraction becomes $\frac{7\sqrt{55}-7\sqrt{15}}{8}$. It is now only necessary to extract *two* square roots, and divide by *one* figure instead of *twenty-one*. The mechanical labor is therefore reduced full *one half*. Moreover, we have tables which give the square roots of all numbers up to 10,000. With the aid of such a table, it would only be necessary to subtract the square root of 15 from the square root of 55, and take $\frac{7}{8}$ of the difference. Thus

$$\begin{array}{r}
\sqrt{55} = 7.4161985 \\
\sqrt{15} = 3.8729833 \\
\hline
3.5432152 \\
7 \\
\hline
8|24.8025064 \\
\hline
\end{array}$$

$$\frac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}} = 3.1003133$$

1. Compute the value of the expression $\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}$.

2. Compute the value of the expression $\frac{\sqrt{3}}{2\sqrt{8}+3\sqrt{5}-7\sqrt{2}}$.

3. Compute the value of the expression $\frac{9+2\sqrt{10}}{9-2\sqrt{10}}$.

CASE XII.

(176.) *To free an equation from radical quantities.*

This may generally be done by successive involutions. For this purpose, we first free the equation from fractions. If there is but one radical expression, we bring that to stand alone on one side of the equation, and involve the whole equation to a power denoted by the index of the radical.

Ex. 1. Free the equation

$$\frac{a+\sqrt{2ax+x^2}}{a} = b$$

from radical quantities.

Clearing of fractions, and transposing a we obtain

$$\sqrt{2ax+x^2}=ab-a.$$

The square of this equation is

$$2ax+x^2=a^2b^2-2a^2b+a^2,$$

which is free from radical quantities.

Ex. 2. Free the equation

$$x+\sqrt{a^2+x^2}=\frac{2a^2}{\sqrt{a^2+x^2}}$$

from radical quantities.

If the equation contains two radical expressions, combined with other terms which are rational, it will generally be best to bring one of the radicals to stand alone on one side of the equation before involution. One of the radicals will thus be made to disappear, and the remaining radical may be exterminated as in the former case.

Ex. 3. Free the equation

$$\sqrt{a+x}+\sqrt{b+y}=c$$

from radical quantities.

Transposing one of the radicals we obtain

$$\sqrt{a+x}=c-\sqrt{b+y}.$$

Squaring we have

$$a+x=c^2-2c\sqrt{b+y}+b+y.$$

Transposing so as to bring the radical to stand alone,

$$2c\sqrt{b+y}=c^2+b+y-a-x,$$

which may be freed from radicals by squaring a second time.

Sometimes the two radicals may be of such a form that it is best to bring both to one member of the equation before involution.

When an equation contains several radical quantities, it may generally be freed from them by successive involutions, but the best mode of procedure can only be determined by trial.

Ex. 4. Free the equation

$$\sqrt{2x+7}+\sqrt{3x-18}=\sqrt{7x+1}.$$

from radical quantities.

When an equation contains a fraction involving radical quantities in both numerator and denominator, it is sometimes best to render the denominator rational by Case XI.

Ex. 5. Free the equation

$$\frac{\sqrt{x}+\sqrt{x-a}}{\sqrt{x}-\sqrt{x-a}}=\frac{an^2}{x-a}$$

from radical quantities.

Multiply both terms of the first fraction by $\sqrt{x}+\sqrt{x-a}$, and we have

$$\frac{(\sqrt{x}+\sqrt{x-a})^2}{x-(x-a)}=\frac{an^2}{x-a}.$$

$$\text{or } (\sqrt{x}+\sqrt{x-a})^2=\frac{a^2n^2}{x-a}.$$

Extracting the square root

$$\sqrt{x}+\sqrt{x-a}=\frac{an}{\sqrt{x-a}}$$

clearing of fractions

$$\sqrt{x^2-ax}+x-a=an,$$

which is easily freed from radicals.

Ex. 6. Free the equation

$$\frac{x+\sqrt{x}}{x-\sqrt{x}}=\frac{x^2-x}{4}$$

from radical quantities.

Ex. 7. Free the equation

$$\frac{x-\sqrt{x+1}}{x+\sqrt{x+1}}=\frac{5}{11}$$

from radical quantities.

Ex. 8. Free the equation

$$\frac{\sqrt{a^2-x^2}-\sqrt{b^2+x^2}}{\sqrt{a^2-x^2}+\sqrt{b^2+x^2}}=\frac{m}{n}$$

from radical quantities.

(177.) The different rules which have been given for the calculation of radicals, are exact, so long as we treat of *absolute numbers;* but are subject to some modifications when we consider *imaginary expressions*, such as $\sqrt{-3}$, $\sqrt{-a}$, etc.

Let it be required, for example, to determine the product of $\sqrt{-a}$ by $\sqrt{-a}$.

By the rule given in Art. 165,

$$\begin{aligned}\sqrt{-a}\times\sqrt{-a}&=\sqrt{-a\times-a}\\&=\sqrt{+a^2}.\end{aligned}$$

But $\sqrt{+a^2}=\pm a$, so that there is apparently a doubt as to the sign with which a ought to be affected in order to answer the question. However, the true result is $-a$; because any quantity must be equal to the square of its square root.

Now $\sqrt{-a}\times\sqrt{-a}$ is the same thing as $(\sqrt{-a})^2$, and consequently is equal to $-a$.

Next let it be required to determine the product of $\sqrt{-a}$ by $\sqrt{-b}$.

By the rule in Art. 165,

$$\begin{aligned}\sqrt{-a}\times\sqrt{-b} &= \sqrt{-a\times -b}\\ &= \sqrt{+ab}\\ &= \pm\sqrt{ab}.\end{aligned}$$

The true result however is $-\sqrt{ab}$, for we have according to Art. 148

$$\sqrt{-a}=\sqrt{a}.\sqrt{-1}$$
$$\sqrt{-b}=\sqrt{b}.\sqrt{-1}.$$

Hence

$$\begin{aligned}\sqrt{-a}\times\sqrt{-b} &= \sqrt{ab}\,(\sqrt{-1})^2\\ &= \sqrt{ab}\;\times -1\\ &= -\sqrt{ab}.\end{aligned}$$

According to this principle, we shall find for the different powers of $\sqrt{-1}$ the following results.

$\sqrt{-1}=\sqrt{-1}$, the first power.

$(\sqrt{-1})^2=-1$, the second power.

$(\sqrt{-1})^3=(\sqrt{-1})^2.\sqrt{-1}$

$\quad=-\sqrt{-1}$, the third power.

$(\sqrt{-1})^4=(\sqrt{-1})^2\times(\sqrt{-1})^2$

$\quad=-1\times-1$

$\quad=+1$, the fourth power.

Since the four following powers will be found by multiplying $+1$ by the first, the second, the third, and the fourth powers, we shall again find for the four next powers

$$+\sqrt{-1},\,-1,\,-\sqrt{-1},\,+1\,;$$

so that all the powers of $\sqrt{-1}$ will form a repeating cycle of these four terms.

Whenever the student is at a loss to determine the product of two imaginary quantities, it is best to resolve each of them into two factors; one the square root of a positive quantity, and the other $\sqrt{-1}$. Art. 148.

Thus let it be required to multiply $\sqrt{-9}$ by $\sqrt{-4}$.

$$\sqrt{-9}=3\sqrt{-1}$$
$$\sqrt{-4}=2\sqrt{-1}.$$

Therefore $\sqrt{-9}\times\sqrt{-4}=3\sqrt{-1}\times 2\sqrt{-1}$
$$=6\sqrt{(-1)^2}$$
$$=-6.$$

Examples. Multiply $1+\sqrt{-1}$ by $1-\sqrt{-1}$.

Multiply $\sqrt{18}$ by $\sqrt{-2}$.

SECTION XII.

EQUATIONS OF THE SECOND DEGREE.

(178.) According to Art. 96, *quadratic equations, or equations of the second degree, are those in which the highest power of the unknown quantity is a square.*

Quadratic equations are divided into *two classes.*

I. Equations which involve only the square of the unknown quantity. These are termed *pure quadratics.* Of this description are the equations

$$ax^2 = b\,;\ 3x^2 + 12 = 150 - x^2\,;\ \text{etc.}$$

They are sometimes called *quadratic equations of two terms,* because by transposition and reduction they can always be exhibited under the general form.

$$ax^2 = b.$$

II. Equations which involve both the square and the first power of the unknown quantity. These are termed *affected* or *complete quadratics.* Of this description are the equations

$$ax^2 + bx = c\,;\ x^2 - 10x = 7\,;\ \frac{5x^2}{6} - \frac{x}{2} + \frac{3}{4} = 8.$$

They are sometimes called quadratic equations of *three terms,* because by transposition and reduction they can always be exhibited under the general form

$$ax^2+bx=c.$$

Solution of pure quadratics containing one unknown quantity.

(179.) The solution of the equation

$$ax^2=b$$

presents no difficulty. Dividing each member by a, it becomes

$$x^2=\frac{b}{a}.$$

$$\text{Whence } x=\pm\sqrt{\frac{b}{a}}$$

If $\frac{b}{a}$ be a particular number, either integral or fractional, we can extract its square root either exactly or approximately by the rules of arithmetic.

It is to be remarked that since the square both of $+m$ and $-m$ is $+m^2$, so in like manner the square of $+\sqrt{\frac{b}{a}}$ and that of $-\sqrt{\frac{b}{a}}$ are both $+\frac{b}{a}$. Hence the above equation is susceptible of *two* solutions, or has *two roots;* that is, there are two quantities which when substituted for x in the original equation will render the two members identical. These are

$$x=+\sqrt{\frac{b}{a}} \text{ and } x=-\sqrt{\frac{b}{a}}.$$

For substitute each of these values in the original equation $ax^2=b$, it becomes

$$a\times(+\sqrt{\frac{b}{a}})^2=b, \text{ or } a\times\frac{b}{a}=b; \text{ i. e. } b=b.$$

and $a \times (-\sqrt{\frac{b}{a}})^2 = b$, or $a \times \frac{b}{a} = b$; i. e. $b = b$.

EXAMPLE I.

Find the values of x which satisfy the equation

$$4x^2 - 7 = 3x^2 + 9.$$

Transposing terms, $4x^2 - 3x^2 = 9 + 7$.

Reducing $x^2 = 16$.

Extracting the square root

$$x = \pm\sqrt{16} = \pm 4.$$

Hence the two values of x are $+4$ and -4, and they may both be verified by substitution in the original equation.

Thus, taking the first value we have

$$4 \times (+4)^2 - 7 = 3 \times (+4)^2 + 9$$
$$4 \times 16 - 7 = 3 \times 16 + 9$$
$$\text{i. e. } 57 = 57.$$

Taking the second value of x, we have

$$4 \times (-4)^2 - 7 = 3 \times (-4)^2 + 9$$
$$4 \times 16 - 7 = 3 \times 16 + 9 \text{ as before.}$$

From the preceding examples we deduce the following

RULE.

Reduce the equation to the form $ax^2 = b$; *then divide by the coefficient of* x^2, *and extract the square root of both members of the equation.*

Ex. 2. Given $x^2 - 17 = 130 - 2x^2$ to find the values of x.

By transposition $3x^2 = 147$,

therefore $x^2 = 49$

and $x = \pm 7$.

Ex. 3. Given $x^2 + ab = 5x^2$ to find the values of x.

By transposition $ab = 4x^2$

therefore $\pm\sqrt{ab} = 2x$

and $x = \dfrac{\pm\sqrt{ab}}{2}$.

Ex. 4. Given $x + \sqrt{a^2 + x^2} = \dfrac{2a^2}{\sqrt{a^2 + x^2}}$ to find the values of x.

Clearing of fractions $x\sqrt{a^2 + x^2} + a^2 + x^2 = 2a^2$.

By transposition $x\sqrt{a^2 + x^2} = a^2 - x^2$.

Squaring both sides, $a^2x^2 + x^4 = a^4 - 2a^2x^2 + x^4$,

therefore $3a^2x^2 = a^4$

and $3x^2 = a^2$

$$x^2 = \frac{a^2}{3}$$

therefore $x = \pm\dfrac{a}{\sqrt{3}}$.

Ex. 5. Given $\dfrac{x^2 + 3x - 7}{x + 2 + \dfrac{18}{x}} = 1$ to find the values of x.

Ans. $x = + 3$.

Ex. 6. Given $3x^2 - 4 = 28 + x^2$ to find the values of x.

Ex. 7. Given $x^2 + 5x = 5x + 4225$ to find the values of x.

Ex. 8. Given $\dfrac{x^2}{3} - 3 + \dfrac{5}{12}x^2 = \dfrac{7}{24} - x^2 + \dfrac{299}{24}$ to find the values of x.

PROBLEMS.

Prob. 1. What two numbers are those whose sum is to the greater as 10 to 7; and whose sum multiplied by the less produces 270?

Let $10x =$ their sum.
Then $7x =$ the greater number,
and $3x =$ the less
Whence $30x^2 = 270$,
and $x^2 = 9$
therefore $x = \pm 3$,

and the numbers are ± 21 and ± 9.

Prob. 2. What two numbers are those whose sum is to the greater as m to n; and whose sum multiplied by the less is equal to a?

$$\text{Ans. } \pm\sqrt{\frac{an^2}{m(m-n)}} \text{ and } \pm\sqrt{\frac{a(m-n)}{m}}.$$

Prob. 3. Find three numbers in the ratio of $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{4}$; the sum of whose squares is 724.

Prob. 4. Find three numbers in the ratio of m, n and p; the sum of whose squares is equal to a.

Ans.

$$\pm\sqrt{\frac{am^2}{m^2+n^2+p^2}};\ \pm\sqrt{\frac{an^2}{m^2+n^2+p^2}};\ \text{and } \pm\sqrt{\frac{ap^2}{m^2+n^2+p^2}}$$

Prob 5. Two travelers A and B set out to meet each other, A leaving Hartford at the same time that B left New York. They traveled the direct road, and on meeting it appeared that A had traveled 18 miles more than B; and that A could have gone B's

Prob. 6. V! V -1

journey in $15\frac{3}{4}$ hours; but B would have been 28 hours in performing A's journey. What was the distance between Hartford and New York?

Ans. 126 miles.

Prob. 6. From two places at an unknown distance, two bodies A and B move towards each other, A going a miles more than B. A would have described B's distance in n hours, and B would have described A's distance in m hours. What was the distance of the two places from each other?

$$\text{Ans. } a \times \frac{\sqrt{m} + \sqrt{n}}{\sqrt{m} - \sqrt{n}}.$$

Prob. 7. Divide the number 49 into two such parts, that the quotient of the greater divided by the less, may be to the quotient of the less divided by the greater, as $\frac{4}{3}$ to $\frac{3}{4}$.

Prob. 8. Divide the number a into two such parts, that the quotient of the greater divided by the less, may be to the quotient of the less divided by the greater, as m to n.

$$\text{Ans. } \frac{a\sqrt{m}}{\sqrt{m} + \sqrt{n}} \text{ and } \frac{a\sqrt{n}}{\sqrt{m} + \sqrt{n}}.$$

Prob. 9. In a court there are two square grass plats; a side of one of which is 10 yards longer than a side of the other; and their areas are as 25 to 9. What are the lengths of the sides?

Prob. 10. There are two squares whose areas are as m to n, and a side of one exceeds a side of the other by a. What are the lengths of the sides?

$$\text{Ans. } \frac{a\sqrt{m}}{\sqrt{m} - \sqrt{n}} \text{ and } \frac{a\sqrt{n}}{\sqrt{m} - \sqrt{n}}.$$

Prob. 11. What number is that, the third part of whose square being subtracted from 20, leaves a remainder equal to 8?

Prob. 12. What number is that, the mth part of whose square being subtracted from a, leaves a remainder equal to b?

Ans. $\pm\sqrt{m(a-b)}$.

Prob. 13. A vintner draws a certain quantity of wine out of a full vessel that holds 256 gallons; and then filling the vessel with water, draws off the same quantity of liquor as before, and so on for four draughts, when there were only 81 gallons of pure wine left. How much wine did he draw each time?

Ans. 64, 48, 36 and 27 gallons.

Prob. 14. A number a is diminished by the nth part of itself, this remainder is diminished by the nth part of itself and so on to the fourth remainder, which is equal to b. Required the number subtracted the first time?

Ans. $a-\sqrt[4]{ba^3}$.

Prob. 15. Two workmen A and B were engaged to work for a certain number of days at different rates. At the end of the time, A who had played 4 of those days had 75 shillings to receive; but B who had played 7 of those days received only 48 shillings. Now had B only played 4 days, and A played 7 days, they would have received exactly alike. For how many days were they engaged?

Ans. 19 days.

Prob. 16. A person employed two laborers, allowing them different wages. At the end of a certain number of days, the first who had played m days received a shillings; and the second who had played n days received b shillings. Now if this last had played m days and the other n days, they would both have received the same sum. For how many days were they engaged?

Ans. $\dfrac{n\sqrt{a} \pm m\sqrt{b}}{\sqrt{a} \pm \sqrt{b}}$ days.

(180.) *Solution of complete quadratics containing one unknown quantity.*

Take the equation

$$x^2 - 6x + 9 = 1$$

it is required to find the value of x.

Since each member of the equation is a *complete square*, if we extract the square root, we shall obtain a new equation involving only the first power of x, which may be easily solved.

We thus have $x - 3 = \pm 1$,

and by transposition

$$x = 3 \pm 1 = 4 \text{ or } 2.$$

In order to verify these values, substitute each of them in place of x in the given equation. Taking the first value we shall have

$$4^2 - 6 \times 4 + 9 = 1,$$

i. e. $16 - 24 + 9 = 1$, an identical equation.

Taking the second value of x we obtain

$$2^2 - 6 \times 2 + 9 = 1,$$

i. e. $4 - 12 + 9 = 1$, an identical equation.

Hence we see that an affected quadratic equation is readily solved, provided each member of the equation is a complete square. But equations are seldom presented to us under this form. Take for example,

$$x^2 - 6x = -8.$$

The preceding method seems to be inapplicable, because the first member is not a complete square. We may, however, render it a complete square by the addition of 9, which must also be added to the second member to preserve the equality. The equation thus becomes,

$$x^2 - 6x + 9 = 9 - 8 = 1,$$

which is the equation originally proposed.

The peculiar difficulty, then, in resolving complete equations of the second degree, consists in rendering the first member an exact square.

(181.) In order to discover a general method of solution, let us take the equation

$$ax^2 + bx = c,$$

which is the general form of equations of the second degree. We begin by dividing both members by a, the coefficient of x^2. The equation then becomes

$$x^2 + \frac{bx}{a} = \frac{c}{a}.$$

For convenience let us put $p = \frac{b}{a}$, and $q = \frac{c}{a}$; we then have

$$x^2 + px = q.$$

We have seen that if we can by any transformation render the first member of this equation the perfect square of a binomial, we can reduce the equation to one of the first degree by extracting the square root.

But we know that the square of a binomial $x + a$, or $x^2 + 2ax + a^2$, is composed of the square of the first term, plus twice the product of the first term by the second, plus the square of the second term.

Hence, considering $x^2 + px$ as the first two terms of the square of a binomial, and consequently px as twice the product of the first term of the binomial by the second, it is evident that the second term of this binomial must be $\frac{p}{2}$, for $2 \times \frac{p}{2} \times x = px$.

In order, therefore, that the above expression may be rendered a perfect square, we must add to it the square of this second term $\frac{p}{2}$, that is, the square of half the coefficient of the first power of x; it thus becomes

$$x^2 + px + \frac{p^2}{4},$$

which is the square of $x + \frac{p}{2}$. But since we have added $\frac{p^2}{4}$ to the left hand member of the equation, in order that the equality between the two members may not be destroyed, we must add the same quantity to the right hand member also; the equation thus transformed will become

$$x^2 + px + \frac{p^2}{4} = q + \frac{p^2}{4}.$$

Extracting the square root

$$x + \frac{p}{2} = \pm\sqrt{q + \frac{p^2}{4}}.$$

$$\text{Whence } x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$$

We affix the double sign $\pm\sqrt{q+\frac{p^2}{4}}$, because the square both of $+\sqrt{q+\frac{p^2}{4}}$, and also of $-\sqrt{q+\frac{p^2}{4}}$ is $+(q+\frac{p^2}{4})$, and every quadratic equation must therefore have two roots.

(182.) From the preceding discussion we deduce the following general

RULE FOR THE SOLUTION OF A COMPLETE QUADRATIC EQUATION.

1. *Transpose all the known quantities to one side of the equation, and all the terms involving the unknown quantity to the other side, and reduce the equation to the form* $ax^2+bx=c$.

2. *Divide each side of the equation by the coefficient of* x^2, *and add to each member the square of half the coefficient of the first power of* x.

3. *Extract the square root of both sides, and the equation will be reduced to one of the first degree, which may be solved in the usual manner.*

EXAMPLE 1.

Reduce the equation $x^2-10x=-16$.

Completing the square by adding to each side the square of half the coefficient of the second term,

$$x^2-10x+25=25-16=9.$$

Extracting the root $x-5=\pm 3$.

Hence, $x=5\pm 3$

$$\begin{cases} x=5+3=8 \\ x=5-3=2. \end{cases}$$

Thus x has two values, either 8 or 2. To verify them, substitute in the original equation, and we shall have

$8^2 - 10 \times 8 = -16$ *i. e.* $64 - 80 = -16$,
also $2^2 - 10 \times 2 = -16$ *i. e.* $4 - 20 = -16$,
both of which are identical equations.

EXAMPLE 2.

Reduce the equation $x^2 + 6x = -8$.
Completing the square $x^2 + 6x + 9 = 9 - 8 = 1$.
Extracting the root $x + 3 = \pm 1$.
Hence $x = -3 \pm 1$

$$\begin{cases} x = -3 + 1 = -2 \\ x = -3 - 1 = -4. \end{cases}$$

Proof. $(-2)^2 + 6 \times -2 = -8$ *i. e.* $4 - 12 = -8$,
also $(-4)^2 + 6 \times -4 = -8$ *i. e.* $16 - 24 = -8$.

Hence x has two values both negative. In verifying them it is to be observed that the square of -2 is $+4$, and -2 multiplied by $+6$ gives -12.

EXAMPLE 3.

Reduce the equation $x^2 + 16x = -60$.
Completing the square $x^2 + 16x + 64 = 64 - 60 = 4$.
Extracting the root $x + 8 = \pm 2$.
Hence $x = -8 \pm 2 = -6$ or -10.

EXAMPLE 4.

Reduce the equation $x^2 + 6x = 27$.
Completing the square $x^2 + 6x + 9 = 27 + 9 = 36$.
Extracting the root $x + 3 = \pm 6$.
Hence $x = -3 \pm 6 = +3$ or -9.

EXAMPLE 5.

Reduce the equation $x^2 - 2x = 24$.
Here $x = 1 \pm 5 = +6$ or -4.

EXAMPLE 6.

Reduce the equation $x^2 - 6x = -10$.

Completing the square $x^2 - 6x + 9 = 9 - 10 = -1$.

Hence $x = 3 \pm \sqrt{-1}$.

Here both values of x are *imaginary*.

EXAMPLE 7.

Reduce the equation $x^2 - 6x = -9$.

Completing the square $x^2 - 6x + 9 = 9 - 9 = 0$.

Extracting the root $x - 3 = \pm 0$.

Hence $x = 3 \pm 0$.

Here the two values of x are equal to each other.

Ex. 8. Given $2x^2 + 8x - 20 = 70$ to find x.

Ans. $x = 5$ or -9.

Ex. 9. Given $3x^2 - 3x + 6 = 5\frac{1}{3}$ to find x.

Ans. $x = \frac{2}{3}$ or $\frac{1}{3}$.

(183.) The Rule given on page 172 for solving a quadratic equation is applicable to all cases; nevertheless a modification of this method is sometimes preferable.

The object is to render the first member of the equation a complete square. After the equation has been reduced to the form

$$ax^2 + bx = c$$

the square may be completed, by *multiplying the equation by four times the coefficient of* x^2, *and adding to both sides the square of the coefficient of* x.

Thus the above equation multiplied by $4a$ becomes

$$4a^2x^2 + 4abx = 4ac.$$

Adding b^2 to both members, we have

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2.$$

Extracting the square root

$$2ax + b = \pm \sqrt{4ac + b^2}.$$

Transposing b and dividing by $2a$,

$$x = -\frac{b \pm \sqrt{4ac + b^2}}{2a},$$

which is the same result as would be obtained by the former Rule; but by this new method we have avoided the introduction of fractions in completing the square.

When $a = 1$, the above Rule becomes, *Multiply the equation by four, and add to both members the square of the coefficient of* x.

Ex. 10. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 20\frac{1}{2} = 42\frac{2}{3}$ to find x.

Ans. $x = 7$, or $-6\frac{1}{3}$.

Ex. 11. Given $x^2 - 8x + 10 = 19$ to find x.
Ex. 12. Given $x^2 - x - 40 = 170$ to find x.
Ex. 13. Given $3x^2 + 2x - 9 = 76$ to find x.
Ex. 14. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{3}{8} = 8$ to find x.

Ex. 15. Given $3x - \frac{6x^2 - 40}{2x - 1} - \frac{3x - 10}{9 - 2x} = 2$ to find x.

We must begin with clearing this equation of fractions, which is done by multiplying by the denominators; we thus obtain
$-12x^3 + 60x^2 - 27x + 12x^3 - 54x^2 + 360 - 80x - 6x^2 + 23x - 10 = 40x - 8x^2 - 18.$

Here the two terms containing x^3 balance each other, and uniting similar terms we obtain

$$8x^2 - 124x = -368.$$

Dividing by 8, $x^2 - 15\frac{1}{2}x = -46.$

Completing the square $x^2 - 15\frac{1}{2}x + \left(\frac{31}{4}\right)^2 = -46 + \left(\frac{31}{4}\right)^2 = \frac{225}{16}.$

Extracting the root $x - \frac{31}{4} = \pm \frac{15}{4}$

$$x=\frac{31}{4}\pm\frac{15}{4}=11\tfrac{1}{2}, \text{ or } 4.$$

Ex. 16. Given $\frac{90}{x}-\frac{90}{x+1}-\frac{27}{x+2}=0$ to find x.

(184.) The preceding rules will enable us to solve not only quadratic equations, but all equations which can be reduced to the form

$$x^{2n}+px^n=q,$$

that is, *all equations which contain only two powers of the unknown quantity, and in which one of the exponents is double of the other.*

For if in the above equation we assume $y=x^n$, then $y^2=x^{2n}$, and it becomes

$$y^2+py=q.$$

Solving this according to the rule

$$x_n=y=-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}}.$$

Extracting the *n*th root of both sides

$$x=\sqrt[n]{-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}}}.$$

EXAMPLE 1.

Given $x^4-25x^2=-144$ to find the values of x.

Assume $x^2=y$, the above becomes

$$y^2-25y=-144.$$

Whence $y=16$ or 9.

But since $x^2=y$, $x=\pm\sqrt{y}$.

Therefore $x=\pm\sqrt{16}$ or $\pm\sqrt{9}$.

Thus x has four values, viz. $+4, -4, +3, -3$.

To verify these values.

1st value; $(+4)^4-25\times(+4)^2=-144$, *i.e.* $256-400=-144$.

2d value; $(-4)^4-25\times(-4)^2=-144$, *i.e.* $256-400=-144$.

3d value ; $(+3)^4-25\times(+3)^2=-144$, *i.e.* $81-225=-144$.
4th value ; $(-3)^4-25\times(-3)^2=-144$, *i.e.* $81-225=-144$.

EXAMPLE 2.

Given $x^4-13x^2=-36$ to find the values of x.

Assume $$x^2=y,$$

then $$y^2-13y=-36$$

$$y^2-13y+\left(\frac{13}{2}\right)^2=\left(\frac{13}{2}\right)^2-36=\frac{25}{4}$$

$$y=\frac{13}{2}\pm\frac{5}{2}=9 \text{ or } 4.$$

Therefore $$x=\pm 3 \text{ or } \pm 2.$$

EXAMPLE 3.

Given $x^4-7x^2=8$.

Assuming $x^2=y$, we have

$$y^2-7y=8.$$

Whence $$y=8 \text{ or } -1.$$

Therefore $x=\pm\sqrt{8}$ or $\pm\sqrt{-1}$, the two last of which roots are *imaginary*.

EXAMPLE 4.

Given $x^6-2x^3=48$ to find x.

Assuming $x^3=y$, the above becomes

$$y^2-2y=48.$$

Whence $$y=8 \text{ or } -6.$$

But since $x^3=y$, therefore $x=\sqrt[3]{y}$.

Hence two of the roots of the above equation are 2 and $-\sqrt[3]{6}$; the remaining four roots cannot be determined by this process.

EXAMPLE 5.

Let $$2x-7\sqrt{x}=99.$$

In this case assume $\sqrt{x} = y$, the equation becomes

$$2y^2 - 7y = 99.$$

Whence $$y = 9 \text{ or } -\frac{11}{2}.$$

But since $\sqrt{x} = y$, therefore $x = y^2$.

Whence $$x = 81 \text{ or } \frac{121}{4}.$$

EXAMPLE 6.

Let $$\sqrt{x+12} + \sqrt[4]{x+12} = 6.$$

Assume $x + 12 = y$, the equation then becomes

$$y^{\frac{1}{2}} + y^{\frac{1}{4}} = 6,$$

which evidently belongs to the same class as the previous examples. Completing the square, we shall have

$$y^{\frac{1}{4}} = 2 \text{ or } -3.$$

Raising both sides of the equation to the fourth power,

$$y = 16 \text{ or } 81.$$

Therefore $$x \text{ or } y - 12 = 4 \text{ or } 69.$$

EXAMPLE 7.

Let $$2x^2 + \sqrt{2x^2 + 1} = 11.$$

Adding 1 to each member of the equation, it becomes

$$2x^2 + 1 + \sqrt{2x^2 + 1} = 12.$$

Assume $$2x^2 + 1 = y, \text{ then}$$

$$y + y^{\frac{1}{2}} = 12.$$

Completing the square and solving, we find

$$y^{\frac{1}{2}} = 3 \text{ or } -4.$$

$$i.\ e.\ \sqrt{2x^2+1} = 3 \text{ or } -4,$$

therefore 2[illegible]$1 = 9$ or 16

$$x^2 = 4 \text{ or } \frac{15}{2}.$$

Hence $$x = +2,\ -2,\ +\sqrt{\frac{15}{2}},\ -\sqrt{\frac{15}{2}}.$$

It may be remarked that it is generally unnecessary to substitute y, which has been done in the preceding examples simply for the sake of perspicuity.

Ex. 8. Given $x^4 + 4x^2 = 12$ to find x.

Ans. $x = \pm\sqrt{2}$.

Ex. 9. Given $x^6 - 8x^3 - 513 = 0$ to find x.

Ans. $x = 3$.

Ex. 10. Given $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$ to find x.

Ans. $x = 243$ or $\sqrt[3]{(-28)^5}$.

Ex. 11. Given $\frac{1}{2}x^6 - \frac{1}{4}x^3 = -\frac{1}{32}$ to find x.

Ex. 12. Given $2x^{\frac{2}{3}} + 3x^{\frac{1}{3}} = 2$ to find x.

Ex. 13. Given $x^4 - 12x^3 + 44x^2 - 48x = 9009$ to find x.

This equation may be expressed as follows:

$$(x^2 - 6x)^2 + 8(x^2 - 6x) = 9009.$$

Ex. 14. Given $\frac{1}{2}x - \frac{1}{3}\sqrt{x} = 22\frac{1}{6}$ to find x.

Ex. 15. Given $\sqrt{10+x} - \sqrt[4]{10+x} = 2$ to find x.

Ex. 16. Given $2x^4 - x^2 + 96 = 99$ to find x.

Ex. 17. Given $x^6 + 20x^3 - 10 = 59$ to find x.

Ex. 18. Given $3x^{2n} - 2x^n + 3 = 11$ to find x.

Ex. 19. Given $x^2 - x\sqrt{3} = x - \frac{1}{2}\sqrt{3}$ to find x.

Ex. 20. Given $\sqrt{1 + x - x^2} - 2(1 + x - x^2) = \frac{1}{9}$ to find x.

(185.) We have seen that an equation of the form $ax^2 + bx = c$ has *two roots*, or that there are two quantities which when substituted for x in the original equation will render the two members identical. In like manner we shall find that every equation of the third degree has *three roots;* an equation of the fourth degree has *four roots;* and in general *an equation has as many roots as it has dimensions.*

The above method of solving the equation $x^{2n} + px^n = q$, will not always give us *all* of the roots, and we must have recourse to different processes to obtain the remaining roots. The subject will be resumed in Section XX.

PROBLEMS PRODUCING QUADRATIC EQUATIONS.

Prob. 1. It is required to find two numbers, such that their difference shall be 8, and their product 240.

Let x = the least number.

Then will $x + 8$ = the greater.

And by the question $x\,(x + 8) = x^2 + 8x = 240$

$x = 12$ the less number,

$x + 8 = 20$ the greater.

Proof. $20 - 12 = 8$, the first condition,

$20 \times 12 = 240$, the second condition.

Prob. 2. The Receiving Reservoir at Yorkville is a rectangle, 60 rods longer than it is broad, and its area is 5500 square rods. Required its length and breadth?

Prob. 3. What two numbers are those whose difference is $2a$, and product b?

Ans. $a \pm \sqrt{b + a^2}$, and $-a \pm \sqrt{b + a^2}$.

Prob. 4. It is required to divide the number 60 into two such parts that their product shall be 864.

Let $x =$ the greater part.
Then will $60 - x =$ the less.
And by the question $x(60 - x) = 60x - x^2 = 864$.

The parts are 36 and 24. Ans.

Prob. 5. In a parcel which contains 24 coins of silver and copper, each silver coin is worth as many pence as there are copper coins, and each copper coin is worth as many pence as there are silver coins, and the whole is worth 18 shillings. How many are there of each?

Prob. 6. What two numbers are those whose sum is $2a$, and product b?

Ans. $a + \sqrt{a^2 - b}$, and $a - \sqrt{a^2 - b}$.

Prob. 7. It is required to find two numbers such that their sum shall be 10, and the sum of their squares 58.

Let $x =$ the greater of the two numbers.
Then will $10 - x =$ the less.
And $x^2 + (10 - x)^2 = 2x^2 - 20x + 100 = 58$

$$x^2 - 10x = -21$$
$$x^2 - 10x + 25 = 4$$
$$x = 5 \pm 2 = 7 \text{ or } 3.$$

The two values of x are the required numbers whose sum is 10. It will be observed we put x to represent the *greater* number, whereas we find it may equal the less as well as the greater. The reason is that we have here imposed a condition which does not enter into the equation. If x represent *either* of the required numbers, then $10 - x$ will represent the *other*, and hence the values of x found by solving the equation should give the less as well as the greater number. Beginners are very apt thus in the statement of a problem to impose conditions which do not appear in the equation.

The preceding example, and all others of the same class may be solved without completing the square. Thus

Let x represent the half difference of the two numbers.

Then according to the principle on page 81,
$5+x$ will represent the greater of the two numbers,
$5-x$ " the less "

The square of $5+x$ is $25+10x+x^2$
" $5-x$ $25-10x+x^2$

The sum is $50 + 2x^2$, which according to the problem $= 58$
Hence $2x^2 = 8$,
or $x^2 = 4$
$x = \pm 2$.

Therefore $5+x=7$ the greater number,
$5-x=3$ less "

Prob. 8. A laborer dug two trenches whose united length was 26 yards, for 356 shillings, and the digging of each of them cost as many shillings per yard as there were yards in its length. What was the length of each?

Prob. 9. What two numbers are those whose sum is $2a$, and the sum of their squares is $2b$?

Ans. $a+\sqrt{b-a^2}$, and $a-\sqrt{b-a^2}$.

Prob. 10. Two persons A and B set out at the same time from two different places, C and D, and traveled towards each other. On meeting it appeared that A had gone 30 miles more than B, and according to the rate at which they travel, A calculates that he can reach the place D in 4 days, and that B can arrive at the place C in 9 days. What is the distance between C and D?

Ans. 150 miles.

Prob. 11. In the preceding problem, let d be the distance which A had traveled more than B; a the time which A requires to travel the remaining distance to D, and b the time which B requires in order to travel the remaining distance to c. What expression will give the distance between C and D?

Ans. $\dfrac{d(\sqrt{b}+\sqrt{a})}{\sqrt{b}-\sqrt{a}}$.

Prob. 12. A person bought a number of sheep for 80 dollars, and if he had bought four more for the same money, he would have paid one dollar less for each. How many did he buy?

Let x represent the number of sheep.

Then will $\dfrac{80}{x}$ be the price of each.

And $\dfrac{80}{x+4}$ = the price of each, if he had bought four more for the same money.]

But by the question

$$\frac{80}{x}=\frac{80}{x+4}+1.$$

Solving this equation, we obtain

$x=16$. Ans.

Prob. 13. A person bought a number of sheep for a dollars. If he had bought $2b$ more for the same money, he would have paid c dollars less for each. How many did he buy?

Ans. $-b\pm\sqrt{\dfrac{2ab+b^2c}{c}}$.

Prob. 14. It is required to find three numbers such that the product of the first and second may be 15, the product of the first and third 21, and the sum of the squares of the second and third 74.

Prob. 15. Required to find three numbers such that the product of the first and second may be a, the product of the first and third b, and the sum of the squares of the second and third c.

Ans. $\sqrt{\frac{a^2+b^2}{c}}$, $a\sqrt{\frac{c}{a^2+b^2}}$, $b\sqrt{\frac{c}{a^2+b^2}}$.

Prob. 16. The sum of two numbers is 16, and the sum of their cubes 1072. What are those numbers?

Ans. 7 and 9.

Prob. 17. The sum of two numbers is $2a$, and the sum of their cubes is $2b$. To find the numbers.

Ans. $a+\sqrt{\frac{b-a^3}{3a}}$ and $a-\sqrt{\frac{b-a^3}{3a}}$.

Prob. 18. Two magnets, whose powers are as 4 to 9, are placed at a distance of 20 inches from each other. It is required to find, on the line which joins their centres, the point where a needle would be equally attracted by both; admitting that the intensity of magnetic attraction varies inversely as the square of the distance.

Ans. $\left\{\begin{array}{l}\text{8 inches from the weakest magnet} \\ \text{or } -40 \text{ " " }\end{array}\right.$

Prob. 19. Two magnets, whose powers are as m to n, are placed at a distance of a feet from each other. It is required to find, on the line which joins their centres, the point which is equally attracted by both.

Ans. The distance from the magnet m is $\frac{a\sqrt{m}}{\sqrt{m}\pm\sqrt{n}}$.

" " n is $\frac{a\sqrt{n}}{\sqrt{m}\pm\sqrt{n}}$.

Prob. 20. A set out from C towards D and traveled 6 miles an hour. After he had gone 45 miles, B set out from D towards C, and went every hour $\frac{1}{20}$ of the entire distance; and after he had traveled as many hours as he went miles in one hour, he met A. Required the distance of the places C and D.

Ans. Either 100 miles or 180 miles.

Prob. 21. A set out from C towards D and traveled a miles per hour. After he had gone b miles, B set out from D towards C, and went every hour $\frac{1}{n}$th of the entire distance; and after he had traveled as many hours as he went miles in one hour, he met A. Required the distance of the places C and D.

$$\text{Ans. } n\left(\frac{n-a}{2} \pm \sqrt{\left(\frac{n-a}{2}\right)^2 - b.}\right)$$

(186.) *On the solution of quadratic equations containing two unknown quantities.*

An equation containing two unknown quantities is said to be of the *second degree, when it involves terms in which the sum of the exponents of the unknown quantities is equal to 2, but never exceeds 2.* Thus

$$3x^2 - 4x + y^2 = 25$$
$$7xy - 4x + y = 40,$$

are equations of the second degree.

The general solution of two equations of the second degree containing two unknown quantities, depends upon the solution of an equation of the fourth degree containing one unknown quantity. Hence the principles already established are not sufficient to enable us to solve *all* equations of this description. Yet there are many particular cases in which they may be reduced either to pure or affected quadratics, and the roots determined in the ordinary manner.

EXAMPLE I.

Given $\left\{\begin{array}{l} x^2 + y^2 = 65 \\ xy \qquad = 28 \end{array}\right\}$ to find the values of x and y.

Multiply the second equation by 2, we have

$$2xy = 56.$$

Adding and subtracting this equation successively from the first,

$$x^2 + 2\,xy + y^2 = 121.$$

And

$$x^2 - 2xy + y^2 = 9.$$

Extracting the square roots of each of these equations

$$x + y = \pm 11,$$

and
$$x - y = \pm\ 3.$$

Adding or subtracting these, we shall have

$$x = 7 \text{ or } -7$$
$$y = 4 \text{ or } -4.$$

(187.) It will sometimes facilitate the operation to substitute for one of the unknown quantities, the product of the other by a third unknown quantity; which method may be applied to advantage whenever the sum of the dimensions of the unknown quantities is the same in every term of the equation. See Article 30.

Ex. 2. Given $\left\{ \begin{array}{l} x^2 + xy \ = 56 \\ xy + 2y^2 = 60 \end{array} \right\}$ to find the values of x and y.

Here, agreeably to the above observation, let $x = vy$, then

$$v^2y^2 + vy^2 = 56$$
$$vy^2 + 2y^2 = 60.$$

Whence from the first of these equations $y^2 = \dfrac{56}{v^2 + v}$,

and from the second $y^2 = \dfrac{60}{v + 2}$.

Therefore by equating the right hand members of these two expressions, we shall have

$$\frac{60}{v+2} = \frac{56}{v^2+v},$$

$$\text{or } 60v^2 + 60v = 56v + 112.$$

Whence $$v^2 + \frac{v}{15} = \frac{28}{15}.$$

Hence by completing the square

$$v = -\frac{1}{30} \pm \sqrt{\frac{1}{900} + \frac{28}{15}} = -\frac{1}{30} \pm \frac{41}{30} = \frac{4}{3} \text{ or } -\frac{7}{5}.$$

And consequently by the former part of the process

$$y^2 = \frac{60}{v+2} = \frac{60}{\frac{4}{3}+2} = 18; \text{ whence } y = \sqrt{18} = 3\sqrt{2}.$$

$$\text{or } y^2 = \frac{60}{2-\frac{7}{5}} = \frac{60}{\frac{3}{5}} = 100; \text{ whence } y = \pm 10,$$

$$\text{and } x = vy = \frac{4}{3} \times 3\sqrt{2} = \pm 4\sqrt{2},$$

$$\text{or } x = -\frac{7}{5} \times 10 = \mp 14.$$

Hence $$y = \pm 10,\ x = \mp 14.$$

The values of x are read *minus or plus* 14, while those of y are read *plus or minus* 10; showing that when the value of y is $+10$, x must be taken -14 in order to satisfy the original equations; and when y is taken -10, x will be equal to $+14$.

(188.) A solution may sometimes be shortened by substituting for the unknown quantities, the sum and difference of two other quantities; which method may be used when the unknown quantities in each equation are similarly involved.

Ex. 3. Given $\left\{\begin{aligned} \frac{x^2}{y} + \frac{y^2}{x} &= 18 \\ x + y &= 12 \end{aligned}\right\}$ to find the values of x and y.

Here according to the above observation assume

$$x = z + v,$$
$$\text{and } y = z - v.$$

Then by adding these two equations together, we shall have

$$x + y = 2z = 12 \text{ or } z = 6.$$

Also since $x = 6 + v$, $y = 6 - v$, and by the first equation $x^3 + y^3 = 18xy$, we shall obtain by substitution

$$(6 + v)^3 + (6 - v)^3 = 18\,(6 + v)\,(6 - v),$$

or by involving the two parts of the first member, and multiplying those of the second

$$432 + 36v^2 = 648 - 18v^2.$$

Whence
$$54v^2 = 216$$
$$v^2 = 4$$
$$\text{or } v = + 2.$$

Hence
$$x = z + v = 6 \pm 2 = 8 \text{ or } 4$$
$$y = z - v = 6 \mp 2 = 4 \text{ or } 8.$$

Ex. 4. Given $\left\{\begin{aligned} x - y &= 15 \\ \frac{xy}{2} &= y^3 \end{aligned}\right\}$ to find the values of x and y.

From the second equation $x = 2y^2$.

Substituting this value in the first equation, we have

$$2y^2 - y = 15.$$

Ex. 5. Given $\left\{\begin{aligned} \frac{10x + y}{xy} &= 3 \\ 9y - 9x &= 18 \end{aligned}\right\}$ to find the values of x and y.

(189.) In many cases it may be convenient to solve the equation first, considering one of the quantities as known; when the rules for exterminating unknown quantities may be more easily applied.

Ex. 6. Given $\left\{\begin{array}{c} 4xy = 96 - x^2y^2 \\ x + y = 6 \end{array}\right\}$ to find the values of x and y.

From the first equation

$$x^2y^2 + 4xy = 96.$$

Completing the square, regarding xy as the unknown quantity,

$$x^2y^2 + 4xy + 4 = 100.$$

Whence $\quad xy = 8$ or -12.

Now squaring the second equation

$$\begin{array}{lrl} & x^2 + 2xy + y^2 & = 36, \\ \text{but} & 4xy & = 32 \text{ or } -48 \\ \hline \text{Therefore by subtraction} & x^2 - 2xy + y^2 & = 4 \text{ or } \quad 84, \\ \text{extracting the root} & x - \quad y & = \pm 2 \text{ or } \pm \surd 84; \\ \text{but} & x + \quad y & = 6. \\ \hline \text{Therefore by addition} & 2x & = 6 \pm 2 \text{ or } 6 \pm \surd 84. \\ \text{Whence} & x & = 4 \text{ or } 2, \text{ or } 3 \pm \surd 21, \\ \text{and by subtraction} & 2y & = 6 \mp 2 \text{ or } 6 \mp \surd 84. \\ \text{Hence} & y & = 2 \text{ or } 4 \text{ or } 3 \mp \surd 21. \end{array}$$

Ex. 7. Given $\left\{\begin{array}{c} x^2 + x + y = 18 - y^2 \\ xy = 6 \end{array}\right\}$ to find the valus of x and y.

$$\begin{array}{lr} \text{By transposition} & x^2 + y^2 + x + y = 18, \\ \text{and from the second equation} & 2xy = 12. \\ \hline \end{array}$$

Therefore by addition $x^2 + 2xy + y^2 + x + y = 30,$

$$\text{or } (x + y)^2 + (x + y) = 30.$$

Ex. 8. Given $\left\{\begin{array}{c} x^2 + 2xy + y^2 + 2x = 120 - 2y \\ xy - y^2 = 8 \end{array}\right\}$ to find the values of x and y.

By transposition $(x+y)^2 + 2(x+y) = 120$.

Ex. 9. Given $\left\{\begin{array}{l} x^2 + y^2 - x - y = 78 \\ xy + x + y = 39 \end{array}\right\}$ to find the values of x and y.

Adding twice the second equation to the first

$$x^2 + 2xy + y^2 + x + y = 156,$$
$$\text{or } (x+y)^2 + (x+y) = 156.$$

Ex. 10. Given $\left\{\begin{array}{l} \dfrac{x^2}{y^2} + \dfrac{4x}{y} = \dfrac{85}{9} \\ x - y = 2 \end{array}\right\}$ to find the values of x and y.

In the first equation $\dfrac{x}{y}$ may be treated as the unknown quantity, and the square completed in the usual way.

Ex. 11. Given $\left\{\begin{array}{l} xy + xy^2 = 12 \\ x + xy^3 = 18 \end{array}\right\}$ to find the values of x and y.

From the first equation $x = \dfrac{12}{y(1+y)}$,

and from the second $x = \dfrac{18}{1+y^3}$.

Whence $$\frac{12}{y(1+y)} = \frac{18}{1+y^3},$$

and dividing by $$\frac{6}{1+y}$$

$$\frac{2}{y} = \frac{3}{y^2 - y + 1}.$$

Therefore $$2y^2 - 2y + 2 = 3y,$$
$$\text{or } 2y^2 - 5y = -2.$$

Ex. 12. Given $\left\{\begin{array}{l} \sqrt{\dfrac{3x-2y}{2x}} + \sqrt{\dfrac{2x}{3x-2y}} = 2 \\ x^2 - 18 = x(4y-9) \end{array}\right\}$ to find the values of x and y.

Multiplying the first equation by $\sqrt{\frac{3x-2y}{2x}}$

$$\frac{3x-2y}{2x}+1=2\sqrt{\frac{3x-2y}{2x}}.$$

By transposition $\frac{3x-2y}{2x}-2\sqrt{\frac{3x-2y}{2x}}+1=0.$

Extracting the square root $\sqrt{\frac{3x-2y}{2x}}-1=0.$

Whence $\quad 3x-2y=2x,$

and $\quad x=2y.$

Now substitute this value of x or y in the second equation.

PROBLEMS.

(190.) 1. To divide the number 100 into two such parts, that the sum of their square roots may be 14.

Ans. 64 and 36.

2. To divide the number a into two such parts, that the sum of their square roots may be b.

Ans. $\frac{a}{2}+\frac{b}{2}\sqrt{2a-b^2}$ and $\frac{a}{2}-\frac{b}{2}\sqrt{2a-b^2}$.

3. The sum of two numbers is 7, and the sum of their fourth powers is 641. What are the numbers?

Ans. 2 and 5.

4. The sum of two numbers is $2a$, and the sum of their fourth powers is b. What are the numbers?

Ans. $a+\sqrt{-3a^2+\sqrt{8a^4+\frac{1}{2}b}}$ and $a-\sqrt{-3a^2+\sqrt{8a^4+\frac{1}{2}b}}$.

5. The sum of two numbers is 6, and the sum of their fifth powers is 1056. What are the numbers?

Ans. 2 and 4.

6. The sum of two numbers is $2a$ and the sum of their fifth powers is b. What are the numbers?

Ans.

$$a+\sqrt{\sqrt{\frac{b}{10a}+\frac{4a^4}{5}}-a^2} \text{ and } a-\sqrt{\sqrt{\frac{b}{10a}+\frac{4a^4}{5}}-a^2}.$$

7. What two numbers are those whose product is 120, and if the greater be increased by 8, and the less by 5, the product of the two numbers thus obtained shall be 300?

Ans. 12 and 10; or 16 and 7.5.

8. What two numbers are those whose product is a, and if the greater be increased by b, and the less by c, the product of the two numbers thus obtained shall be d?

$$\text{Ans.}\ \frac{f}{2}\pm\sqrt{\frac{f^2}{4}-\frac{ab}{c}} \text{ and } \frac{a}{\frac{f}{2}+\sqrt{\frac{f^2}{4}-\frac{ab}{c}}}$$

$$\text{where } f=\frac{d}{c}-\frac{a}{c}-b.$$

9. To find two numbers, such that their sum, their product and the difference of their squares may be all equal to one another.

$$\text{Ans. } \frac{3}{2}+\sqrt{\frac{5}{4}} \text{ and } \frac{1}{2}+\sqrt{\frac{5}{4}}.$$

10. Divide the number 100 into two such parts, that their product and the difference of their squares may be equal to each other.

Ans. 38.196 and 61.804.

11. Divide the number a into two such parts, that their product and the difference of their squares may be equal to each other.

Ans. $\frac{3a}{2}-\sqrt{\frac{5a}{4}}$ and $-\frac{a}{2}+\sqrt{\frac{5a^2}{4}}$.

(191.) *Discussion of the general equation of the second degree.*

We have seen that every equation of the second degree may be reduced to the form

$$x^2+px=q,$$

where p and q are known quantities, positive or negative, integral or fractional, numerical or algebraical.

The values of x in this equation are

$$x=-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}.$$

$$x=-\frac{p}{2}-\sqrt{q+\frac{p^2}{2}}.$$

Hence,

PROPERTY I.

Every equation of the second degree has two roots, and only two.

A *root* of an equation is such a number as, being substituted for the unknown quantity, will satisfy the equation.

This principle has been often exemplified in the preceding pages. Two values have uniformly been found for x, although both values may not be applicable to the problem which furnishes the equation.

(192.) If we multiply

$$x+\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}=0,$$

$$\text{by } x+\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}=0,$$

we shall obtain $x^2+px-q=0,$

which was the equation originally proposed.

Hence,

PROPERTY II.

Every equation of the second degree, whose roots are a and b, may be resolved into the two factors, $x-a$ and $x-b$.

Ex. 1. Thus the equation

$$x^2-10x+16=0,$$

may be resolved into the factors $\left\{\begin{array}{l} x-8=0, \\ x-2=0, \end{array}\right.$

where 8 and 2 are the roots of the given equation.

It is also obvious that if a is a root of an equation of the second degree, this equation must be divisible by $x-a$. Thus the preceding equation is divisible by $x-8$, giving the quotient $x-2$.

Ex. 2. The roots of the equation

$$x^2+6x+8=0,$$

are -2 and -4. Resolve it into its factors.

Ex. 3. The roots of the equation

$$x^2+6x-27=0,$$

are $+3$ and -9. Resolve it into its factors.

Ex. 4. The roots of the equation

$$x^2-2x-24=0,$$

are $+6$ and -4. Resolve it into its factors.

(193.) If we add together the two values of x in the general equation of the second degree, the radical parts having opposite signs disappear and we obtain

$$-\frac{p}{2}-\frac{p}{2}=-p.$$

Hence,

PROPERTY III.

The algebraic sum of the two roots is equal to the coefficient of the second term of the equation, taken with a contrary sign.

Thus, in Example 1, page 172,

$$x^2-10x=-16,$$

the two roots are 8 and 2, whose sum is $+10$, the coefficient of x taken with a contrary sign.

In the equation

$$x^2+6x=-8,$$

the two roots are -2 and -4.

In the equation

$$x^2+16x=-60,$$

the two roots are -6 and -10.

(194.) If we multiply together the two values of x, (since the product of the sum and difference of two quantities is equal to the difference of their squares), we obtain

$$\frac{p^2}{4}-\left(q+\frac{p^2}{4}\right)=-q.$$

Hence,

PROPERTY IV.

The product of the two roots is equal to the second member of the equation, taken with a contrary sign.

Thus, in the equation

$$x^2-10x=-16,$$

the product of the two roots 8 and 2 is $+16$, which is equal to the second member of the equation taken with a contrary sign.

So also in the equation

$$x^2+6x=27,$$

whose two roots are $+3$ and -9, their product is -27.

REAL AND IMAGINARY VALUES.

(195.) Since $\frac{p^2}{4}$, being a square, is positive for all real values of p, it follows that the expression $q+\frac{p^2}{4}$ can only be rendered negative by q.

When q is positive, or when q is negative and numerically less than $\frac{p^2}{4}$, then will $q+\frac{p^2}{4}$ be positive, and consequently $\sqrt{q+\frac{p^2}{4}}$ will be *real*. This happens in nearly all the preceding examples.

When q is negative, and numerically greater than $\frac{p^2}{4}$, then $q+\frac{p^2}{4}$ will be negative, and consequently $\sqrt{q+\frac{p^2}{4}}$ will be *imaginary*. This happens in Ex. 6, p. 174.

CASE I.

When $\sqrt{q+\frac{p^2}{4}}$ *is real.*

1. When p is negative, and $\frac{p}{2}$ is numerically greater than $\sqrt{q+\frac{p^2}{4}}$, *both values of x will be real and positive.*

This happens in the equation

$$x^2-6x=-8,$$

whose two roots are 4 and 2.

Also in the equation

$$x^2-10x=-16,$$

whose two roots are 8 and 2.

2. When p is positive, and $\frac{p}{2}$ is numerically greater than $\sqrt{q+\frac{p^2}{4}}$, *both values of x will be real and negative.*

This happens in the equation

$$x^2+6x=-8,$$

whose two roots are -2 and -4.

Also in the equation

$$x^2+16x=-60,$$

whose two roots are -6 and -10.

3. When $\frac{p}{2}$ is numerically less than $\sqrt{q+\frac{p^2}{4}}$, *both values of x will be real, the one positive and the other negative.*

This happens in the equation

$$x^2+6x=27,$$

whose roots are $+3$ and -9.

Also in the equation

$$x^2-2x=24,$$

whose roots are $+6$ and -4.

CASE II.

(196.) *When $\sqrt{q+\frac{p^2}{4}}$ is imaginary.*

In this case, *both values of x are imaginary.*

This happens in the equation

$$x^2-6x=-10,$$

whose roots are $3\pm\sqrt{-1}$.

We will now show that in this case the conditions of the question are *incompatible* with each other, and therefore the values of x *ought* to be imaginary.

For this purpose we must first establish the following principle.

The greatest product which can be obtained by dividing a number into two parts and multiplying them together, is the square of half that number.

Let $p=$ the given number,
and $d=$ the difference of the parts.

Then from page 81, $\frac{p}{2}+\frac{d}{2}=$ the greater part,

$\frac{p}{2}-\frac{d}{2}=$ the less part,

and $\frac{p^2}{4}-\frac{d^2}{4}=$ the product of the parts.

Now it is plain that this expression will be the greatest possible when $d=0$, that is, $\frac{p^2}{4}$ is the greatest product, which is the square of $\frac{p}{2}$, half the given number.

For example, let 12 be the number to be divided.

We have $12=1+11$; and $11\times1=11$
$12=2+10$; " $10\times2=20$
$12=3+9$; " $9\times3=27$
$12=4+8$; " $8\times4=32$
$12=5+7$; " $7\times5=35$
$12=6+6$; " $6\times6=36$.

We here see that the smaller the difference of the two parts, the greater the product; and this product is greatest when the two parts are equal.

Now, in the equation

$$x^2 - px = -q,$$

p is the sum of the roots and q their product. Therefore q can never be greater than $\frac{p^2}{4}$.

If, then, any problem furnishes an equation in which q is negative, and greater than $\frac{p^2}{4}$, we infer that the conditions of the question are incompatible with each other.

Thus, in the example

$$x^2 - 6x = -10,$$

$\frac{p^2}{4} = 9$, which is numerically less than q. The equation requires us to divide the number 6 into two parts whose product shall be 10, which is an impossibility; and, accordingly, in solving the equation, we obtain *imaginary* values for x.

Hence, *an imaginary root indicates an absurdity in the proposed question which furnished the equation.*

Suppose it is required to divide 8 into two such parts, that their product shall be 18.

Let $x =$ one of the parts,
$8 - x =$ the other.

Then by the conditions

$$x(8 - x) = 18.$$

Whence $$x^2 - 8x = -18.$$

This equation, solved by the usual method, gives

$x = 4 \pm \sqrt{-2}$, an imaginary expression.

Hence we infer that it is *impossible* to find two numbers whose sum is 8 and product 18. This is obvious from the Proposition above demonstrated, from which it appears that 16 is the greatest product which can be obtained by decomposing 8 into two parts, and multiplying them together.

(197.) When q is negative, and numerically equal to $\frac{p^2}{4}$, the radical part of both values of x becomes zero, and both values of x reduce to $-\frac{p}{2}$. *The two roots are then said to be equal.*

Thus, in the equation

$$x^2 - 6x = -9,$$

the two roots are 3 and 3.

We say that in this case the equation has two roots, because it is the product of the two factors, $x - 3 = 0$, and $x - 3 = 0$.

DISCUSSION OF PARTICULAR PROBLEMS.

(198.) In discussing particular problems which involve equations of the second degree, we meet with all the different cases which are presented by equations of the first degree, and some peculiarities besides. We may therefore have,

1. Positive values of x.
2. Negative values.
3. Values of the form of $\frac{0}{A}$.
4. Values of the form of $\frac{A}{0}$.
5. Values of the form of $\frac{0}{0}$.

All these different cases are presented by Prob. 19, p. 184, when we make different suppositions upon the values of a, m, and n; but we need not dwell upon them here.

The peculiarities exhibited by equations of the second degree are,

6. Double values of x.
7. Imaginary values.

We will consider the last two cases.

(199.) *Double values of the unknown quantity.*

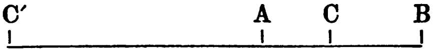
We have seen that every equation of the second degree has two roots. Sometimes both of these values are applicable to the problem which furnishes the equation. Thus, in Prob. 20, p. 184, we obtain either 100 or 180 miles for the distance between the places C and D.

Let E represent the situation of A when B sets out on his journey. Then, if we suppose C D equals 100 miles, E D will equal 55 miles, of which A will travel 30 miles (being 6 miles an hour for 5 hours); and B will travel 25 miles (being 5 miles an hour for 5 hours).

If we suppose C D equals 180 miles, E D will equal 135 miles, of which A will travel 54 miles (being 6 miles an hour for 9 hours); and B will travel 81 miles (being 9 miles an hour for 9 hours).

This problem, therefore, admits of two positive answers, both equally applicable to the question.

In Prob. 18, p. 184, one of the values of x is positive, and the other negative.



Let the weakest magnet be placed at A, and the strongest at B; then C will represent the situation of a needle equally attracted by both magnets. According to the first value, the distance $AC = 8$ inches, and $CB = 12$. Now at the distance of 8 inches, the power of the weakest magnet will be represented by $\frac{4}{8^2}$; and at the distance of 12 inches, the power of the other magnet will be represented by $\frac{9}{12^2}$, and these two powers are equal, for

$$\frac{4}{8^2}=\frac{9}{12^2}.$$

But there is another point, C′, which equally satisfies the conditions of the question; and this point is 40 inches to the left of A, and therefore 60 inches to the left of B; for

$$\frac{4}{40^2}=\frac{9}{60^2}.$$

(200.) *Imaginary values of the unknown quantity.*

We have seen that an imaginary root indicates an absurdity in the proposed question which furnished the equation.

In several of the preceding problems, the values of x become imaginary in particular cases.

When will the values of x in Prob. 6, p. 181, be imaginary?

Ans. When $b > a^2$.

What is the absurdity involved in this supposition?

Ans. It is absurd to suppose that the product of two numbers can be greater than the square of half their sum.

When will the values of x in Prob. 9, p. 182, be imaginary?

Ans. When $a^2 > b$; or $(2a)^2 > 4b$.

What is the absurdity of this supposition?

Ans. The square of the sum of two numbers cannot be greater than twice the sum of their squares.

When will the values of x in Prob. 17, p. 184, be imaginary?

Ans. When $a^3 > b$; or $(2a)^3 > 8b$.

What is the absurdity of this supposition?

Ans. The cube of the sum of two numbers cannot be greater than four times the sum of their cubes.

When will the values of x in Prob. 12, p. 168, be imaginary, and what is the absurdity of this supposition?

SECTION XIII.

RATIO AND PROPORTION.

(201.) NUMBERS may be compared in two ways: either by their *difference*, or by their *quotient*. We may inquire *how much* one quantity is greater than another; or, *how many times* the one contains the other. One is called Arithmetical, and the other Geometrical Ratio.

The difference between two numbers is called their *Arithmetical Ratio*. Thus, the arithmetical ratio of 9 to 7 is 9 — 7, or 2; and if a and b designate two numbers, their arithmetical ratio is represented by $a - b$.

Numbers are more generally compared by means of quotients; that is, by inquiring how many times one number contains another. The quotient of one number divided by another is called their *Geometrical Ratio*. The term Ratio, when used without any qualification, is always understood to signify a geometrical ratio, and we shall confine our attention to ratios of this description.

(202.) By the *ratio* of two numbers, then, we mean *the quotient which arises from dividing one of these numbers by the other.*

Thus, the ratio of 12 to 4 is represented by $\frac{12}{4}$, or 3;

The ratio of 5 to 2 is $\frac{5}{2}$, or 2.5.

The ratio of 1 to 3 is $\frac{1}{3}$, or .333.

We here perceive that the value of a ratio cannot always be expressed exactly; but, by taking a sufficient number of terms of the decimal, we can approach as nearly as we please to the true value.

If a and b designate two numbers, the ratio of a to b is the quotient arising from dividing a by b, and may be represented by writing them $a : b$, or $\frac{a}{b}$. The first term, a, is called the *antecedent* of the ratio, the last term, b, is called the *consequent* of the ratio.

Hence it appears that the theory of ratios is identified with the theory of fractions, and a ratio may be considered as *a fraction whose numerator is the antecedent, and whose denominator is the consequent.*

(203.) When the antecedent of a ratio is greater than the consequent, the ratio is called *a ratio of greater inequality;* as, $\frac{5}{3}$, $\frac{12}{4}$. When the antecedent is less than the consequent, it is called *a ratio of less inequality;* as, $\frac{2}{3}$, $\frac{5}{9}$. When the antecedent and consequent are equal, it is called *a ratio of equality;* as, $\frac{3}{3}$, $\frac{8}{8}$. It is plain that a ratio of equality may always be represented by unity.

(204.) When the corresponding terms of two or more simple ratios are multiplied together, the ratios are said to be *compounded.* Thus, the ratio of $\frac{a}{b}$, compounded with the ratio of $\frac{c}{d}$, becomes $\frac{ac}{bd}$.

When a ratio is compounded with itself, the result is called *a duplicate ratio.* Thus, the duplicate ratio of $\frac{2}{3}$ is $\frac{4}{9}$; the duplicate ratio of $\frac{a}{b}$ is $\frac{a^2}{b^2}$.

A ratio compounded of three equal ratios is called *a triplicate ratio.* Thus, the triplicate ratio of $\frac{2}{3}$ is $\frac{8}{27}$; the triplicate ratio of $\frac{a}{b}$ is $\frac{a^3}{b^3}$.

The ratio of the *square roots* of two quantities is called a *sub-*

duplicate ratio. Thus, the subduplicate ratio of $\frac{4}{9}$ is $\frac{2}{3}$; the subduplicate ratio of $\frac{a}{b}$ is $\frac{\sqrt{a}}{\sqrt{b}}$.

The ratio of the *cube roots* of two quantities is called a *subtriplicate ratio.* Thus, the subtriplicate ratio of $\frac{8}{27}$ is $\frac{2}{3}$; the subtriplicate ratio of $\frac{a}{b}$ is $\frac{\sqrt[3]{a}}{\sqrt[3]{b}}$.

(205.) *If the terms of a ratio are both multiplied, or both divided by the same quantity, the value of the ratio remains unchanged.*

The ratio of a to b is represented by the fraction $\frac{a}{b}$, and the value of a fraction is not changed if we multiply or divide both numerator and denominator by the same quantity. Thus,

$$\frac{a}{b}=\frac{ma}{mb}=\frac{\frac{a}{n}}{\frac{b}{n}};$$

or,

$$a:b=ma:mb=\frac{a}{n}:\frac{b}{n}.$$

(206.) Ratios are compared with each other by reducing the fractions which represent them to a common denominator.

In order to ascertain whether the ratio of 2 to 7 is greater or less than that of 3 to 8, we represent these ratios by the fractions $\frac{2}{7}$ and $\frac{3}{8}$, and reduce them to a common denominator. They thus become

$$\frac{16}{56} \text{ and } \frac{21}{56};$$

and, since the latter of these is the greatest, we infer that the ratio of 2 to 7 is less than the ratio of 3 to 8.

(207.) A ratio of *greater inequality is diminished*, and a ratio of *less inequality is increased*, by adding the same quantity to both terms.

Thus,
$$\frac{3}{2} > \frac{3+1}{2+1}, \text{ or } \frac{4}{3};$$
$$\frac{2}{3} < \frac{2+1}{3+1}, \text{ or } \frac{3}{4}.$$

To prove the proposition generally, let $\frac{a}{b}$ represent any ratio, and let x be added to each of its terms. The two ratios will then be

$$\frac{a}{b} \text{ and } \frac{a+x}{b+x},$$

which, reduced to a common denominator, become

$$\frac{ab+ax}{b(b+x)}, \qquad \frac{ab+bx}{b(b+x)}.$$

If $a > b$, that is, if $\frac{a}{b}$ is a ratio of greater inequality, then, since ax is greater than bx, the first of these fractions is greater than the second, and therefore $\frac{a}{b}$ is *diminished* by the addition of the same quantity to each of its terms.

But if $a < b$, that is, if $\frac{a}{b}$ is a ratio of less inequality, then, since ax is less than bx, the first of the above fractions is less than the second, and therefore $\frac{a}{b}$ is *increased* by the addition of the same quantity to each of its terms.

(208.) If, in a series of ratios, the consequent of each is the antecedent of the following ratio, then *the ratio of the first antecedent to the last consequent is equal to that which is compounded of all the intervening ratios.*

Let the proposed ratios be

$$\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \frac{d}{e}, \frac{e}{f}.$$

Compounding them by Art. 204, we obtain

$$\frac{abcde}{bcdef},$$

which, by dividing by $bcde$, reduces to

$$\frac{a}{f}.$$

PROPORTION.

(209.) *Proportion is an equality of ratios.*

Thus, if a, b, c, d are four quantities, such that a, when divided by b, gives the same quotient as c when divided by d, then a, b, c, d are called *proportionals*, and we say that a is to b as c is to d; and this is expressed by writing them thus:

$$a : b :: c : d,$$

or $$a : b = c : d,$$

or $$\frac{a}{b} = \frac{c}{d}.$$

So, also, 3, 4, 9, 12 are proportionals; that is,

$$3 : 4 :: 9 : 12,$$

or $$\frac{3}{4} = \frac{9}{12}.$$

In ordinary language, the terms *ratio* and *proportion* are confounded with each other. Thus, two quantities are said to be in the proportion of 3 to 5, instead of the ratio of 3 to 5. A ratio subsists between *two* quantities, a proportion only between *four*. Ratio is the quotient arising from dividing one quantity by another; *two equal ratios form a proportion.*

(210.) In the proportion

$$a : b :: c : d,$$

a, b, c, d are called the *terms* of the proportion. The first and last terms are called the *extremes*, the second and third the *means*. The first term is called the *first antecedent*, the second term the *first consequent*, the third term the *second antecedent*, and the fourth term the *second consequent.*

(211.) When the second and third terms of a proportion are identical, this quantity is called a *mean proportional* between the other two. Thus, if we have three quantities, a, b, c, such that

$$a : b :: b : c,$$

then b is called a *mean proportional* between a and c, and c is called a *third proportional* to a and b.

If, in a series of proportional magnitudes, each consequent is identical with the next antecedent, these quantities are said to be in *continued proportion*. Thus, if we have a, b, c, d, e, f, such that

$$a : b :: b : c :: c : d :: d : e :: e : f,$$

or $$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \frac{e}{f},$$

the quantities a, b, c, d, e, f are in continued proportion.

(212.) If four quantities are proportional, *the product of the extremes is equal to the product of the means.*

Let $$a : b :: c : d,$$

or $$\frac{a}{b} = \frac{c}{d}.$$

Multiplying these equals by bd, the expression becomes

$$\frac{abd}{b} = \frac{bcd}{d},$$

or $$ad = bc.$$

Thus, if $$3 : 4 :: 9 : 12,$$

then $$3 \times 12 = 4 \times 9.$$

(213.) Conversely, if the product of two quantities is equal to the product of two others, *the first two quantities may be made the extremes, and the other two the means of a proportion.*

Let $$ad = bc.$$

Dividing these equals by bd, the expression becomes

$$\frac{a}{b} = \frac{c}{d}, \text{ or } \frac{c}{d} = \frac{a}{b};$$

that is, $$a : b :: c : d, \text{ or } c : d :: a : b.$$

Thus, if $$3 \times 12 = 4 \times 9;$$

then $$3 : 4 :: 9 : 12,$$

or $$9 : 12 :: 3 : 4.$$

(214.) This proposition is called the *test* of proportions, and any change may be made in the form of a proportion which is consistent with the application of this test. In order, then, to decide whether four quantities are proportional, we compare the product of the extremes with the product of the means.

Thus, to determine whether 5, 6, 7, 8 are proportional, we multiply 5 by 8, and obtain 40. Multiplying 6 by 7, we obtain 42. As these two products are *not* equal, we conclude that the numbers 5, 6, 7, 8 are *not proportional.*

Again, take the numbers 5, 6, 10, 12. The product of 5 by 12 is 60, and the product of 6 by 10 is also 60. Hence the numbers are proportional; that is,

$$5 : 6 :: 10 : 12.$$

(215.) If three quantities are in continued proportion, *the product of the extremes is equal to the square of the mean.*

If $\qquad a : b :: b : c.$

Then, by Art. 212, $\qquad ac = bb = b^2.$

Conversely, if the product of two quantities is equal to the square of a third, *the last quantity will be a mean proportional between the other two.*

Thus, let $\qquad ac = b^2.$

Dividing these equals by bc, we obtain

$$\frac{a}{b} = \frac{b}{c},$$

or $\qquad a : b :: b : c.$

Thus, if $\qquad 4 : 6 :: 6 : 9,$

then $\qquad 4 \times 9 = 6^2.$

And conversely, if $\qquad 4 \times 9 = 6^2,$

then 6 is a mean proportional between 4 and 9.

EXAMPLES.

1. Given the first three terms of a proportion, 24, 15, and 40, to find the fourth term.

2. Given the first three terms of a proportion, $3ab^3$, $4a^2b^2$, and $9a^3b$, to find the fourth term.

3. Given the last three terms of a proportion, $4a^3b^5$, $3a^3b^3$, and $2a^5b$, to find the first term.

4. Given the first, second, and fourth terms of a proportion, $5y^4$, $7x^2y^3$, and $21x^6y$, to find the third term.

5. Given the first, third, and fourth terms of a proportion, 1, $a - b$, and $a^2 - b^2$, to find the second term.

(216.) *Ratios that are equal to the same ratio are equal to each other.*

$$\left.\begin{array}{ll}\text{Let} & a : b :: x : y,\\ \text{and} & c : d :: x : y,\end{array}\right\}\text{ then } a : b :: c : d.$$

For, from the first of these proportions,

$$\frac{a}{b} = \frac{x}{y}.$$

And from the second,

$$\frac{c}{d} = \frac{x}{y}.$$

Therefore,

$$\frac{a}{b} = \frac{c}{d},$$

or

$$a : b :: c : d.$$

(217.) If four quantities are proportional, they will be proportional by *alternation*; that is, *the first will have the same ratio to the third that the second has to the fourth.*

Let $\quad a : b :: c : d,$

Then, also, $\quad a : c :: b : d.$

For, applying the test of Art. 214, we have in each case

$$ad = bc.$$

(218.) If four quantities are proportional, they will be proportional by *inversion*; that is, *the second will have to the first the same ratio that the fourth has to the third.*

Let $\quad a : b :: c : d,$

then will $\quad b : a :: d : c.$

For, applying the test of Art. 214, we have, from the first proportion, $\quad ad = bc,$

and from the second, $\quad bc = ad.$

(219.) If four quantities are proportional, they will be proportional by *composition*; that is, *the sum of the first and second will have to the second the same ratio that the sum of the third and fourth has to the fourth.*

Let $a : b :: c : d$,

then will $a + b : b :: c + d : d$.

For, from the first proportion, $\frac{a}{b} = \frac{c}{d}$.

Add unity to each member, and we have

$$\frac{a}{b} + 1 = \frac{c}{d} + 1, \text{ or } \frac{a+b}{b} = \frac{c+d}{d};$$

that is, $a + b : b :: c + d : d$.

(220.) If four quantities are proportional, they will be proportional by *division*; that is, *the difference of the first and second will have to the second the same ratio that the difference of the third and fourth has to the fourth.*

Let $a : b :: c : d$,

then will $a - b : b :: c - d : d$.

For, from the first proportion, $\frac{a}{b} = \frac{c}{d}$.

Subtract unity from each member, and we have

$$\frac{a}{b} - 1 = \frac{c}{d} - 1, \text{ or } \frac{a-b}{b} = \frac{c-d}{d};$$

that is, $a - b : b :: c - d : d$.

(221.) If four quantities are proportional, they will be proportional by *conversion*; that is, *the first will have to the difference of the first and second the same ratio that the third has to the difference of the third and fourth.*

Let $a : b :: c : d$,

then will $a : a - b :: c : c - d$.

For, by inversion, $b : a :: d : c$; whence $\frac{b}{a} = \frac{d}{c}$.

Subtract each member from unity, and we have

$$1-\frac{b}{a}=1-\frac{d}{c}, \text{ or } \frac{a-b}{a}=\frac{c-d}{c};$$

that is, $a-b : a :: c-d : c$,

or inversely, $a : a-b :: c : c-d$.

(222.) If four quantities are proportional, *the sum of the first and second will have to their difference the same ratio that the sum of the third and fourth has to their difference.*

Let $a : b :: c : d$,

then will $a+b : a-b :: c+d : c-d$.

For, by composition, $a+b : b :: c+d : d$,

and inverting the means, $a+b : c+d :: b : d$.

Also, by division, $a-b : b :: c-d : d$,

and inverting the means, $a-b : c-d :: b : d$.

Hence, by equality of ratios, $a+b : a-b :: c+d : c-d$.

(223.) If four quantities are proportional, *like powers or roots of these quantities will also be proportional.*

Let $a : b :: c : d$,

then will $a^n : b^n :: c^n : d^n$.

For, since $\frac{a}{b}=\frac{c}{d}$, raising each of these equals to the *nth* power, we have

$$\frac{a^n}{b^n}=\frac{c^n}{d^n};$$

that is, $a^n : b^n :: c^n : d^n$,

where n may be either integral or fractional.

(224.) If there is any number of proportional quantities all having the same ratio, *the first will have to the second the same ratio that the sum of all the antecedents has to the sum of all the consequents.*

Let a, b, c, d, e, f be any number of proportional quantities, such that

$$a : b :: c : d :: e : f,$$

then will $a : b :: a+c+e : b+d+f$.

For, by Art. 212, we have $af=be$,

$$ad=bc;$$

also, $ab=ba$.

Therefore $a(b+d+f)=b(a+c+e)$.

Hence, by Art. 213, $a:b::a+c+e:b+d+f$.

(225.) If three quantities are in continued proportion, *the first will have to the third the duplicate ratio of that which it has to the second.*

Let $a:b::b:c$,

Then $a:c::a^2:b^2$.

For, by Art. 212, $ac=b^2$.

Multiplying these equals by a, we have

$$a^2c=ab^2;$$

$$\text{i. e., } a^2\times c=a\times b^2.$$

Hence, by Art. 213, $a:c::a^2:b^2$.

(226.) If four quantities are in continued proportion, *the first will have to the fourth the triplicate ratio of that which it has to the second.*

Let a, b, c, d be four quantities in continued proportion, so that

$$a:b::b:c::c:d,$$

then will $a:d::a^3:b^3$.

For, by Art. 212, we have $ad=bc$,

$$ac=b^2;$$

Also, $ab=ba$.

Therefore, multiplying these equals together, we have

$$a^3(bdc)=b^3(abc),$$

or $a^3\times d=b^3\times a$.

Hence, by Art. 213, $a:d::a^3:b^3$.

(227.) If there are two sets of proportional quantities, *the products of the corresponding terms will be proportional.*

Let $a:b::c:d$,

and $e:f::g:h$.

Then will $ae:bf::cg:dh$.

For, by Art. 212, $ad=bc$,

and $eh=fg$.

Multiplying these equals together, we have

$$ae \times dh = bf \times cg$$

Hence, by Art. 213, $ae : bf :: cg : dh$.

(228.) Three quantities are said to be in *harmonical* proportion when *the first is to the third as the difference between the first and second is to the difference between the second and third.*

Thus, 2, 3, 6 are in harmonical proportion, for

$$2 : 6 :: 3 - 2 : 6 - 3.$$

Let a, b, c be in harmonical proportion, then

$$a : c :: a - b : b - c.$$

Multiplying the extremes and means, and reducing, we have

$$c = \frac{ab}{2a - b},$$

where c is said to be a third harmonical proportional to a and b.

(229.) Four quantities are said to be in harmonical proportion when *the first is to the fourth as the difference between the first and second is to the difference between the third and fourth.*

Thus, 2, 3, 4, 8 are in harmonical proportion, for

$$2 : 8 :: 3 - 2 : 8 - 4.$$

Let a, b, c, d be in harmonical proportion, then

$$a : d :: a - b : c - d.$$

Multiplying the extremes and means, and reducing, we have

$$d = \frac{ac}{2a - b},$$

where d is a fourth harmonical proportional to a, b, c.

(230.) Proportions are often expressed in an *abridged form.* Thus, if A and B represent two sums of money put out for one year at the same rate of interest, then

A : B : : interest of A : interest of B.

This is briefly expressed by saying that the interest *varies as* the principal. A peculiar character $\propto$ is used to denote this relation. Thus, we write

the interest $\propto$ the principal.

One quantity varies *directly* as another, when both increase or diminish together in the same ratio. Thus, in the above example, A varies directly as the interest of A. In such a case either quantity is equal to the other multiplied by some constant number. Thus, if the interest varies as the principal, then the interest *equals* the principal multiplied by a constant quantity, which is the *rate* of interest.

If $$A \propto B, \text{ then } A = mB.$$

If the space (S) described by a falling body varies as the square of the time (T), then

$$S = mT^2,$$

m representing some constant quantity.

(231.) One quantity varies *inversely* as another, when one increases in the same ratio that the other diminishes. Thus, the altitude of a triangle whose area is given, varies inversely as its base.

If one quantity varies inversely as another, the product of the two quantities is *constant*.

Thus, $$na \times \frac{b}{n} = ab.$$

Conversely, if the product of two quantities is constant, then one varies inversely as the other.

Thus, if $$AB = m,$$

then $$A = \frac{m}{B} \propto \frac{1}{B}.$$

For example, if $$xy = 24,$$

then $$x = \frac{24}{y}.$$

If $x = 2$, then $y = 12$,

If $x = 4$, then $y = 6$;

that is, if x is *doubled*, then y is *halved*. The one varies inversely as the other.

(232.) One quantity may vary as the product of several others. Thus, in uniform motion, the space varies as the time multiplied into the velocity. The area of a rectangle varies as the product of its length and breadth.

The weight of a stick of timber varies as its length × its breath × its depth × its density.

If the density is given, then the weight varies as the length × the breadth × the depth.

If the depth also is given, then the weight varies as the length × the breadth.

If the breadth is given, then the weight varies as the length.

Finally, if the length also is given, then the weight is equal to a *constant* quantity.

(233.) One quantity may vary directly as a second, and inversely as a third. Thus, according to the Newtonian law of gravitation, the attraction (G) of a heavenly body varies directly as the quantity of matter (Q), and inversely as the square of the distance (D).

That is, $$G \propto \frac{Q}{D^2}.$$

(234.) *Application of the preceding principles.*

Ex. 1. Given $\left.\begin{array}{r} x+y : x :: 5 : 3, \\ xy = 6, \end{array}\right\}$ to find the values of x and y.

Since $$x+y : x :: 5 : 3.$$

By Art. 220, $$y : x :: 2 : 3.$$

Therefore, $$3y = 2x, \text{ and } y = \frac{2x}{3}.$$

Substituting this value in the second equation, we obtain

$$\frac{2x^2}{3} = 6,$$

and $$x^2 = 9.$$

Therefore, $$x = \pm 3,$$

whence $$y = \pm 2.$$

Ex. 2. Given $\left.\begin{array}{r} x+y : x-y : : 3 : 1, \\ x^3-y^3=56, \end{array}\right\}$ to find x and y.

From the first equation, by Art. 222, we obtain

$$2x : 2y : : 4 : 2;$$

whence, $$x : y : : 2 : 1,$$

and $$x=2y.$$

Substituting this value of x in the second equation, we obtain

$$y=2; \quad x=4.$$

Ex. 3. Given $\left.\begin{array}{r} \overline{x+y}^2 : \overline{x-y}^2 : : 64 : 1, \\ xy=63, \end{array}\right\}$ to find x and y.

By Art. 223, $$x+y : x-y : : 8 : 1.$$

By Art. 222, $$2x : 2y : : 9 : 7,$$

whence $$x : y : : 9 : 7.$$

Therefore, $$x=\frac{9y}{7}.$$

Substituting this value for x in the second equation, we obtain

$$y=\pm 7; \quad x=\pm 9.$$

Ex. 4. Given $\left.\begin{array}{r} x^3-y^3 : \overline{x-y}^3 : : 61 : 1, \\ xy=320, \end{array}\right\}$ to find the values of x and y.

Since $$x^3-y^3 : x^3-3x^2y+3xy^2-y^3 : : 61 : 1.$$

By Art. 220, $$3xy \times (x-y) : \overline{x-y}^3 : : 60 : 1.$$

Hence $$960 : \overline{x-y}^2 : : 60 : 1,$$

and $$16 : \overline{x-y}^2 : : 1 : 1.$$

Therefore, $$x-y=\pm 4.$$

Since $$x^2-2xy+y^2=16.$$

Also, $$4xy=1280.$$

By addition, $$x^2+2xy+y^2=1296.$$

Extracting the root, $$x+y=\pm 36.$$

Hence $$x=\pm 20, \text{ or } \pm 16,$$

$$y+\pm 16, \text{ or } \pm 20.$$

Ex. 5. Given $\left.\begin{array}{r} x^3 - y^3 : x^2y - xy^2 : : 7 : \overset{2}{\cancel{8}}, \\ x + y = 6, \end{array}\right\}$ to find x and y.

Ans. $x = 4$, or 2,
$y = 2$, or 4.

Ex. 6. Given $\left.\begin{array}{r} \sqrt{y} - \sqrt{a - x} = \sqrt{y - x}, \\ \sqrt{y - x} + \sqrt{a - x} : \sqrt{a - x} : : 5 : 2, \end{array}\right\}$ to find x and y.

Ans. $x = \dfrac{4a}{5}$; $y = \dfrac{5a}{4}$.

Ex. 7. Given $x + \sqrt{x} : x - \sqrt{x} : : 3\sqrt{x} + 6 : 2\sqrt{x}$, to find the values of x.

Ans. $x = 9$, or 4.

Ex. 8. What number is that to which, if 1, 5, and 13 be severally added, the first sum shall be to the second as the second to the third? 3

Ex. 9. What number is that to which, if a, b, and c be severally added, the first sum shall be to the second as the second to the third?

Ans. $\dfrac{b^2 - ac}{a - 2b + c}$.

Ex. 10. What two numbers are those whose difference, sum, and product are as the numbers 2, 3, and 5 respectively?

Ex. 11. What two numbers are those whose difference, sum, and product are as the numbers m, n, and p?

Ans. $\dfrac{2p}{n + m}$, and $\dfrac{2p}{n - m}$.

Ex. 12. Find two numbers, the greater of which shall be to the less as their sum to 42, and as their difference to 6. 24

Ex. 13. Find two numbers, the greater of which shall be to the less as their sum to a, and their difference to b.

Ans. $\dfrac{(a + b)^2}{2(a - b)}$, and $\dfrac{a + b}{2}$.

Ex. 14. There are two numbers which are in the ratio of 3 to 2; the difference of whose fourth powers is to the sum of their cubes as 26 to 7. Reouired the numbers. 6 one 4 other

Ex. 15. What two numbers are those which are in the ratio of m to n; the difference of whose fourth powers is to the sum of their cubes as p to q?

Ans. $\frac{mp}{q} \times \frac{m^3+n^3}{m^4-n^4}$, and $\frac{np}{q} \times \frac{m^3+n^3}{m^4-n^4}$.

SECTION XIV.

PROGRESSIONS.

ARITHMETICAL PROGRESSION.

(235.) *An Arithmetical Progression is a series of quantities which increase or decrease by the continued addition or subtraction of the same quantity.*

Thus, the numbers

$$1, 3, 5, 7, 9, 11, \&c.,$$

which are obtained by the addition of 2 to each successive term, form what is called an *increasing* Arithmetical Progression; and the numbers

$$20, 17, 14, 11, 8, 5, \&c.,$$

which are obtained by the subtraction of 3 from each successive term, form what is called a *decreasing* Arithmetical Progression.

(236.) If a represent the first term of an arithmetical progression, and d the common difference, the successive terms of an increasing series will be

$$a, a+d, a+2d, a+3d, a+4d, \&c.$$

The successive terms of a decreasing series will be

$$a, a-d, a-2d, a-3d, a-4d, \&c.$$

Since the coefficient of d in the *second* term is 1, in the *third* term 2, in the *fourth* term 3, and so on, the nth term of the series will be of the form

$$a \pm (n-1)d,$$

which may be called the *last* term when the number of terms is n. Hence, *The last term is equal to the first,* ± *the product of the common difference into the number of terms less one.*

In what follows we shall consider the progression an *increasing* one, since all the results which we obtain can be immediately applied to a decreasing series by changing the sign of d.

If we put l to represent the last term of the series, we shall accordingly have

$$l = a + (n - 1)d.$$

This equation contains four quantities, any one of which may be computed when the other three are known.

(237.) *To find the sum of* n *terms of a series.*

Take any series, and under it set the same terms in an inverted order, thus:

Take the series	1,	3,	5,	7,	9,	11,	13,	15.
and the same series inverted,	15,	13,	11,	9,	7,	5,	3,	1.
The sums are,	16,	16,	16,	16,	16,	16,	16,	16.

The sum of the two series must be double the sum of a single series, and is equal to the sum of the extremes repeated as many times as there are terms.

In order to generalize this method, let S represent the sum of the series,

Then $S = a + \overline{a + d} + \overline{a + 2d} + \overline{a + 3d} + \ldots\ldots\ldots\ldots + l.$

Write the same series in an inverted order, thus:

$$S = l + \overline{l - d} + \overline{l - 2d} + \overline{l - 3d} + \ldots\ldots\ldots + a.$$

Adding the two series together, term by term, we obtain

$$2S = \overline{l + a} + \overline{l + a} + \overline{l + a} + \overline{l + a} + \ldots\ldots\ldots + \overline{l + a}.$$

Represent the number of terms in the series by n; then

$$2S = n(l + a).$$

Hence $$S = \frac{n(l + a)}{2}.$$

Therefore,

The sum of an Arithmetical Progression is equal to half the sum of the two extremes, multiplied by the number of terms.

It also appears from the above, that *the sum of the extremes is equal to the sum of any other two terms equally distant from the extremes.*

(238.) The two equations

$$l = a + (n - 1)d,$$

$$S = \frac{l+a}{2} \cdot n,$$

contain five variable quantities,

$$a,\ l,\ d,\ n,\ S,$$

of which any three being given, the other two may be found. Accordingly, 20 different cases may arise, all of which are solved by combining the formulæ above given. These cases are exhibited in the following table, and should be verified by the student:

No.	Given.	Required.	Formulæ.
1	$a,\ d,\ n$	l	$l = a + (n - 1)d,$
2	$a,\ d,\ S$		$l = -\frac{1}{2}d \pm \sqrt{2dS + (a - \frac{1}{2}d)^2},$
3	$a,\ n,\ S$		$l = \frac{2S}{n} - a,$
4	$d,\ n,\ S$		$l = \frac{S}{n} + \frac{(n-1)d}{2}.$
5	$a,\ d,\ n$	S	$S = \frac{1}{2}n \left\{ 2a + (n - 1)d \right\},$
6	$a,\ d,\ l$		$S = \frac{l+a}{2} + \frac{(l+a)(l-a}{2d},$
7	$a,\ n,\ l$		$S = \frac{l+a}{2} \times n,$
8	$d,\ n,\ l$		$S = \frac{1}{2}n \left\{ 2l - (n - 1)d \right\}.$
9	$a,\ n,\ l$	d	$d = \frac{l-a}{n-1},$
10	$a,\ n,\ S$		$d = \frac{2S - 2an}{n(n-1)},$
11	$a,\ l,\ S$		$d = \frac{(l+a)(l-a)}{2S - l - a},$
12	$n,\ l,\ S$		$d = \frac{2nl - 2S}{n(n-1)}.$

No.	Given.	Required.	Formulæ.
13	d, n, l	a	$a=l-(n-1)d,$
14	d, n, S		$a=\frac{S}{n}-\frac{(n-1)d}{2},$
15	d, l, S		$a=\frac{1}{2}d\pm\sqrt{(l+\frac{1}{2}d)^2-2dS},$
16	n, l, S		$a=\frac{2S}{n}-l.$
17	a, d, l	n	$n=\frac{l-a}{d}+1,$
18	a, d, S		$n=\frac{\sqrt{(2a-d)^2+8dS}-2a+d}{2d},$
19	a, l, S		$n=\frac{2S}{l+a},$
20	d, l, S		$n=\frac{2l+d\pm\sqrt{(2l+d)^2-8dS}}{2d}.$

(239.) **EXAMPLES.**

Ex. 1. Required the sum of 60 terms of an arithmetical progression whose first term is 5, and common difference 10. 18

Ex. 2. Required the number of terms of a progression whose sum is 442, whose first term is 2, and common difference 3. 17

Ex. 3. Required the first term of a progression whose sum is 99, whose last term is 19, and common difference 2. 5

Ex. 4. The sum of a progression is 1455, the first term 5, and the last term 92. What is the common difference? 3

Ex. 5. A body falls 16 feet during the first second, and in each succeeding second 32 feet more than in the one immediately preceding. If it continue falling during the space of 20 seconds, how many feet will it pass over in the last second, and how many in the whole time?

Ex. 6. Required the sum of 101 terms of the series

1, 3, 5, 7, 9, &c.

Ex. 7. Find the last term of the series

1, 3, 5, 7, 9, &c.

Ans. $2n-1$;

that is, *the last term of this series is one less than twice the number of terms.*

Ex. 8. Find the sum of the series

1, 3, 5, 7, 9, &c.

Ans. n^2;

that is, *the sum of the terms of this series is equal to the square of the number of terms.*

Thus,

$$\begin{aligned} 1+3 &= 4=2^2 \\ 1+3+5 &= 9=3^2 \\ 1+3+5+7 &= 16=4^2 \\ 1+3+5+7+9 &= 25=5^2. \end{aligned}$$

Ex. 9. Find the sum of the natural series of numbers

1, 2, 3, 4, 5, &c.,

up to n terms.

Ans. $\frac{n(n+1)}{2}$.

Ex. 10. Find the sum of the even numbers

2, 4, 6, 8, &c.,

up to n terms.

Ans. $n(n+1)$.

Ex. 11. One hundred stones being placed on the ground in a straight line, at the distance of two yards from each other; how far will a person travel who shall bring them one by one to a basket which is placed two yards from the first stone? A. m- 540

Ex. 12. To find m arithmetical means between two given numbers.

In order to solve this problem, we must first find the common difference. The whole number of terms consists of the two extremes, and all the intermediate terms. If, then, m represent the number of means, $m+2$ will be the whole number of terms.

Substituting $m+2$ for n, in formula 9, p. 222, we have

$$d=\frac{l-a}{m+1}=\text{the common difference,}$$

whence the required means are easily obtained by addition.

Ex. 13. Find 6 arithmetical means between 1 and 50.

Ex. 14. Find three numbers in arithmetical progression, the sum of whose squares shall be 1232, and the square of the mean greater than the product of the two extremes by 16. 16 -

Ex. 15. Find three numbers in arithmetical progression, the sum of whose squares shall be a, and the square of the mean greater than the product of the two extremes by b.

$$\text{Ans. } \sqrt{\frac{a-2b}{3}} - \sqrt{b}\,;\ \sqrt{\frac{a-2b}{3}}\,;\ \text{and } \sqrt{\frac{a-2b}{3}} + \sqrt{b}.$$

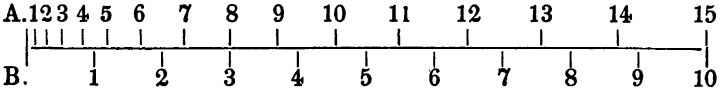
Ex. 16. Find four numbers in arithmetical progression, whose sum is 28, and continued product 585.

Ex. 17. A sets out for a certain place, and travels 1 mile the first day, 2 the second, 3 the third, and so on. In five days afterward, B sets out, and travels 12 miles a day. How long will A travel before he is overtaken by B?

Ans. 8 or 15 days.

d - 1
a - 1
n - x

This is another example of an equation of the second degree, in which the two roots are both positive. The following figure exhibits the daily progress of each traveler. The divisions above the horizontal line represent the distances traveled each day by A; those below the line the distances traveled by B.



It is readily seen from the figure that A is in advance of B until the end of his 8th day, when B overtakes and passes him. After the 12th day, A gains upon B, and passes him on the 15th day, after which he is continually gaining upon B, and could not be again overtaken.

Ex. 18. A goes 1 mile the first day, 2 the second, and so on. B starts a days later, and travels b miles per day. How long will A travel before he is overtaken by B?

$$\text{Ans. } \frac{2b-1 \pm \sqrt{(2b-1)^2 - 8ab}}{2} \text{ days.}$$

In what case would B *never* overtake A?

$$\text{Ans. When } a > \frac{b}{2} - \tfrac{1}{2} + \frac{1}{8b}.$$

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For instance, in the preceding example, if B had started one day later, he could never have overtaken A.

Ex. 19. A traveler set out from a certain place and went 1 mile the first day, 3 the second, 5 the third, and so on. After he had been gone three days, a second sets out, and travels 12 miles the first day, 13 the second, and so on. In how many days will the second overtake the first ?

Ans. in 2 or 9 days.

Let the student illustrate this example by a diagram like the preceding.

GEOMETRICAL PROGRESSION.

(240.) *A Geometrical Progression is a series of quantities, each of which is equal to the product of that which precedes it by a constant number.*

Thus, the series

$$2, 4, 8, 16, 32, \&c.,$$

and

$$81, 27, 9, 3, \&c.,$$

are geometrical progressions. In the former, each number is derived from the preceding by multiplying it by 2, and the series forms an *increasing* geometrical progression. In the latter, each number is derived from the preceding by multiplying it by $\frac{1}{3}$, and the series forms a *decreasing* geometrical progression.

In each of these cases, the common multiplier is called the *common ratio.*

(241.) *To find the last term of the progression.*

Let a represent the first term, and r the common ratio ; then the successive terms of the series will be

$$a, ar, ar^2, ar^3, ar^4, \&c.$$

The exponent of r in the *second* term is 1, in the *third* term is 2, in the *fourth* term 3, and so on ; hence the nth term of the series will be

$$ar^{n-1}.$$

Hence, putting l for the last term, and n the number of terms of the series, we obtain

$$l=ar^{n-1}.$$

That is,

The last term of a geometrical progression is equal to the product of the first term by that power of the ratio whose index is one less than the number of terms.

(242.) *To find the sum of all the terms.*

If we take any geometrical series, and multiply each of its terms by the ratio, a new series will be formed, of which every term except the last will have its corresponding term in the first series. Thus, take the series

$$1, 2, 4, 8, 16, 32,$$

the sum of which we will represent by S, so that

$$S = 1 + 2 + 4 + 8 + 16 + 32.$$

Multiplying each term by 2, we obtain

$$2S = 2 + 4 + 8 + 16 + 32 + 64.$$

The terms of the two series are identical, except the *first* term of the first series and the *last* term of the second series. If, then, we subtract one of these equations from the other, all the remaining terms will disappear, and we shall have

$$2S - S = 64 - 1.$$

In order to generalize this method, let a, ar, ar^2, &c., represent any geometrical series, and S its sum, then

$$S = a + ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-2} + ar^{n-1}.$$

Multiplying this equation by r, we have

$$rS = ar + ar^2 + ar^3 + ar^4 + \ldots\ldots + ar^{n-1} + ar^n.$$

Subtracting the first equation from the second, we obtain

$$rS - S = ar^n - a.$$

Hence $$S = \frac{ar^n - a}{r - 1};$$

or, substituting the value of l already found, we obtain

$$S = \frac{lr - a}{r - 1}.$$

Hence, to find the sum of the terms of a geometrical progression, *Multiply the last term by the ratio, subtract the first term, and divide the remainder by the ratio less one.*

If a series is a decreasing one, and r consequently represents a fraction, it is convenient to change the signs of both numerator and denominator in this expression, which then becomes

$$S=\frac{a-ar^n}{1-r}=\frac{a-lr}{1-r}.$$

(243.) In the two equations

$$l=ar^{n-1},$$
$$S=\frac{lr-a}{r-1},$$

there are five variable quantities,

$$a,\ l,\ r,\ n,\ S,$$

of which *any three* being given, the other two may be found. Accordingly, as in arithmetical progression, 20 different cases may arise, all of which are readily solved, with the exception of those in which n is the quantity sought. The value of n can only be found by the solution of an exponential equation. See Art. 336. These different cases are all exhibited in the following table for convenient reference.

No.	Given.	Required.	Formulæ.
1	a, r, n		$l=ar^{n-1}$,
2	a, r, S		$l=\frac{a+(r-1)S}{r}$,
3	a, n, S		$l(S-l)^{n-1}-a(S-a)^{n-1}=0$,
4	r, n, S		$l=\frac{(r-1)Sr^{n-1}}{r^n-1}$.
5	a, r, n	S	$S=\frac{a(r^n-1)}{r-1}$,
6	a, r, l		$S=\frac{lr-a}{r-1}$,
7	a, n, l		$S=\frac{l^{\frac{n}{n-1}}-a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}}-a^{\frac{1}{n-1}}}$.
8	r, n, l		$S=\frac{l(r^n-1)}{(r-1)r^{n-1}}$.

No.	Given.	Required.	Formulæ.
9	a, n, l	r	$r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}$,
10	a, n, S		$r^n - \frac{S}{a}r + \frac{S-a}{a} = 0$,
11	a, l, S		$r = \frac{S-a}{S-l}$,
12	n, l, S		$r^n - \frac{S}{S-l}r^{n-1} + \frac{l}{S-l} = 0$.
13	r, n, l	a	$a = \frac{l}{r^{n-1}}$,
14	r, n, S		$a = \frac{(r-1)S}{r^n - 1}$,
15	r, l, S		$a = rl - (r-1)S$,
16	n, l, S		$a(S-a)^{n-1} - l(S-l)^{n-1} = 0$.
17	a, r, l	n	$n = \frac{\log. l - \log. a}{\log. r} + 1$,
18	a, r, S		$n = \frac{\log. [a + (r-1)S] - \log. a}{\log. r}$
19	a, l, S		$n = \frac{\log. l - \log. a}{\log. (S-a) - \log. (S-l)} + 1$,
20	r, l, S		$n = \frac{\log. l - \log. [rl - (r-1)S]}{\log. r} + 1$.

EXAMPLES.

Ex. 1. Required the sum of the series

1, 3, 9, 27, &c.,

continued to 12 terms.

Ex. 2. Required the sum of the series

1, 2, 4, 8, 16, &c.,

continued to 14 terms.

Ex. 3. Given the first term 2, the ratio 3, and the number of terms 10, to find the last term.

Ex. 4. Given the first term 1, the last term 512, and the sum of the terms 1023, to find the ratio.

Ex. 5. Given the last term 2048, the number of terms 12, and the ratio 2, to find the first term.

Ex. 6. A person being asked to dispose of his horse, said he would sell him on condition of receiving one cent for the first nail in his shoes, two cents for the second, and so on, doubling the price of every nail to 32, the number of nails in his four shoes. What would the horse sell for at that rate?

Ans. $42949672.95.

(244.) *To find any number of geometrical means between two given numbers.*

In order to solve this problem, it is only necessary to know the *ratio*. If m represent the number of *means*, $m+2$ will be the whole number of *terms*. Substituting $m+2$ for n in formula 9, Art. 243, we obtain

$$r = \sqrt[m+1]{\frac{l}{a}}$$

When the ratio is known, the required means are obtained by continued multiplication.

Ex. 1. Find three geometrical means between 2 and 162.

Ex. 2. Find two geometrical means between 4 and 256.

(245.) *Of decreasing progressions having an infinite number of terms.*

The formula

$$S = \frac{a - ar^n}{1 - r},$$

which represents the sum of n terms of a decreasing series, may be put under the form

$$S = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

In a decreasing progression, since r is a proper fraction, r^n is less than unity, and the larger the number n, the smaller will be the quantity r^n. If, therefore, we take a very large number of terms of the series, the quantity r^n, and consequently the term $\frac{ar^n}{1-r}$, will be very small; and if we take n greater than any assignable number, then $\frac{ar^n}{1-r}$ will be less than any assignable number.

Hence the sum of an infinite series decreasing in geometrical progression is

$$S = \frac{a}{1 - r}.$$

Ex. 1. Find the sum of the infinite series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} +, \&c.$$

Here $a = 1,\ r = \tfrac{1}{2}.$

Therefore, $S = \dfrac{a}{1 - r} = \dfrac{1}{1 - \frac{1}{2}} = 2.$

Ex. 2. Find the sum of the infinite series

$$1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} +, \&c.$$

Ans. $\tfrac{3}{2}$.

Ex. 3. Find the sum of the infinite series

$$1 + \tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{64} +, \&c.$$

Ex. 4. Find the ratio of an infinite progression, whose first term is 1, and the sum of the series $\tfrac{5}{4}$.

Ex. 5. Find the first term of an infinite progression, whose ratio is $\tfrac{1}{10}$, and the sum $\tfrac{2}{3}$.

Ex. 6. Find the first term of an infinite progression, of which the ratio is $\dfrac{1}{n}$, and the sum $\dfrac{n}{n - 1}$.

(246.) PROBLEMS.

Prob. 1. Of four numbers in geometrical progression, the sum of the first and second is 15, and the sum of the third and fourth is 60. Required the numbers.

Let $x,\ xy,\ xy^2,\ xy^3$, be the numbers.

Therefore, $x + xy = 15,$

and $xy^2 + xy^3 = 60.$

Multiplying the first equation by y^2,

$$xy^2 + xy^3 = 15y^2 = 60.$$

Therefore, $y^2 = 4,$

and $y = \pm 2.$

Also, $x \pm 2x = 15.$

Therefore, $x = 5 \text{ or } -15.$

Taking the first value of x, and the corresponding value of y, we obtain the series

$$5,\ 10,\ 20,\ 40;$$

which numbers may be easily verified.

Taking the second value of x, and the corresponding value of y, we obtain the series

$$-15,\ +30,\ -60,\ +120;$$

which numbers also perfectly satisfy the problem understood algebraically. If, however, it is required that the terms of the progression be *positive*, the last value of x would be inapplicable to the problem, though satisfying the algebraic equation.

Prob. 2. There are three numbers in geometrical progression whose sum is 210, and the first exceeds the last by 90. What are the numbers? [illegible]

Prob. 3. There are three numbers in geometrical progression whose continued product is 64, and the sum of their cubes is 584. Required the numbers. [illegible]

Prob. 4. There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the sum of the means as 7 to 3. Required the numbers. [illegible]

Prob. 5. Of four numbers in geometrical progression, the difference between the first and second is 36, and the difference between the third and fourth is 4. What are the numbers?

Prob. 6. Of four numbers in geometrical progression, the sum of the first and third is a, the sum of the second and fourth is b. What are the numbers?

$$\text{Ans. } \frac{a^3}{a^2+b^2},\ \frac{a^2b}{a^2+b^2},\ \frac{ab^2}{a^2+b^2},\ \frac{b^3}{a^2+b^2}.$$

HARMONICAL PROGRESSION.

(247.) *A series of quantities is said to be in harmonical progression when, of any three consecutive terms, the first is to the third as the difference of the first and second is to the difference of the second and third.*

The numbers

$$60,\ 30,\ 20,\ 15,\ 12,\ 10,$$

are in harmonical progression, for

$$60 : 20 :: 60 - 30 : 30 - 20$$
$$30 : 15 :: 30 - 20 : 20 - 15$$
$$20 : 12 :: 20 - 15 : 15 - 12$$
$$15 : 10 :: 15 - 12 : 12 - 10.$$

So, also, the numbers

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}, \text{\&c.},$$

form an harmonical progression.

(248.) *The reciprocals of a series of terms in harmonical progression are in arithmetical progression.*

Thus, the reciprocals of 60, 30, 20, &c., are

$$\tfrac{1}{60}, \tfrac{1}{30}, \tfrac{1}{20}, \tfrac{1}{15}, \tfrac{1}{12}, \tfrac{1}{10},$$

which are respectively equal to

$$\tfrac{1}{60}, \tfrac{2}{60}, \tfrac{3}{60}, \tfrac{4}{60}, \tfrac{5}{60}, \tfrac{6}{60},$$

being an arithmetical progression whose common difference is $\tfrac{1}{60}$.

If six musical strings of equal weight and tension have their lengths in the ratio of the numbers

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6},$$

the second will sound the octave of the first; the third will sound the twelfth; the fourth will sound the double octave; the fifth will sound the eighteenth; and the sixth will sound the third octave of the first. Hence the origin of the term *harmonical* or *musical* proportion.

Let a, b, c be three quantities in harmonical progression, then

$$a : c :: a - b : b - c;$$

whence
$$b = \frac{2ac}{a+c}.$$

That is, *an harmonical mean between two quantities is equal to twice their product divided by their sum.*

SECTION XV.

GREATEST COMMON DIVISOR.—PERMUTATIONS AND COMBINATIONS.

(249.) The greatest common divisor of two or more quantities is the greatest factor which is common to each of the quantities.

PROPOSITION.

The greatest common divisor of two quantities is the same with the greatest common divisor of the least quantity, and their remainder after division.

To prove this principle, let the greatest of the two quantities be represented by A, and the least by B. Divide A by B; let the entire part of the quotient be represented by Q, and the remainder by R. Then, since the dividend must be equal to the product of the divisor by the quotient + the remainder,

$$A = QB + R.$$

Now every number which will divide B will divide QB; and every number which will divide R and QB will divide R + QB or A. That is, every number which is a common divisor of B and R is a common divisor of A and B.

Again, every number which will divide A and B will divide A and QB; it will also divide A — QB or R. That is, every number which is a common divisor of A and B is also a common divisor of B and R. Hence the greatest common divisor of A and B must be the same as the greatest common divisor of B and R.

(250.) To find, then, the greatest common divisor of two quantities, we divide the greater by the less; and the remainder, which is necessarily less than either of the given quantities, is by the last article divisible by the greatest common divisor.

Dividing the preceding divisor by the last remainder, a still smaller remainder will be found, which is divisible by the greatest common divisor; and by continuing this process with each remainder and the preceding divisor, quantities smaller and smaller are found, which are all divisible by the greatest common divisor, until at length the greatest common divisor must be obtained. Hence the following

RULE.

Divide the greater quantity by the less, and the preceding divisor by the last remainder, till nothing remains; the last divisor will be the greatest common measure.

When the remainders decrease to unity, the given quantities have *no* common divisor, and are said to be *incommensurable*, or *prime* to each other.

EXAMPLES.

Ex. 1. What is the greatest common divisor of 372 and 246?

$$\begin{array}{rrrrl}
 & & & 372 \mid & 246 \\
 & & & 246 \mid & 1 \\
 & & 246 \mid & 126, & \text{first Rem.} \\
 & & 126 \mid & 1 & \\
 & 126 \mid & 120, & \text{second Rem.} & \\
 & 120 \mid & 1 & & \\
120 \mid & 6, & \text{third Rem.} & & \\
120 \mid & 20 & & &
\end{array}$$

Here we have continued the operation of division until we obtain 0 for a remainder; the last divisor (6) is the greatest common divisor. Thus, 246 and 372 being each divided by 6, give 41 and 62, and these quotients are *prime* with respect to each other; that is, have no common divisor greater than unity.

Ex. 2. What is the greatest common divisor of

336 and 720?

Ex. 3. What is the greatest common divisor of

918 and 522?

(251.) In the application of this rule to polynomials, some modification may become necessary. It may happen that the first term of the dividend is not divisible by the first term of the divisor. This may arise from the presence of a factor in the divisor which is not found in the dividend, and may therefore be suppressed. For, since the greatest common divisor of two quantities is only the product of their *common* factors, it cannot be affected by a factor of the one quantity which is *not found* in the other.

We therefore suppress in the first polynomial all the factors common to each of its terms. We do the same with the second polynomial, and if the suppressed factors have a common divisor, we reserve it as forming part of the common divisor sought.

But if, after this reduction, the first term of the dividend, when arranged according to the powers of some letter, is not divisible by the first term of the arranged divisor, it follows from the above that we may multiply the dividend by any monomial factor which will render its first term divisible by the first term of the divisor.

EXAMPLES.

Ex. 1. Required the greatest common divisor of

$$x^5+x^3 \text{ and } x^4-1.$$

$$\begin{array}{r|l} x^5+x^3 & x^4-1 \\ \cline{2-2} x^5-x & x \\ \hline x^3+x, & \text{first Rem} \end{array}$$

Suppressing x, we have x^2+1.

$$\begin{array}{r|l} x^4-1 & x^2+1 \\ \cline{2-2} x^4+x^2 & x^2-1 \\ \hline -x^2-1 & \\ -x^2-1. & \end{array}$$

Whence x^2+1 is the greatest common divisor. To verify this result, divide x^5+x^3 by x^2+1, and we obtain x^3; divide x^4-1 by x^2+1, and we obtain x^2-1.

Ex. 2. Required the greatest common divisor of

$$x^3-b^2x \text{ and } x^2+2bx+b^2.$$

$$\begin{array}{l|l} x^3-b^2x & x^2+2bx+b^2 \\ \cline{2-2} x^3+2bx^2+b^2x & x \\ \hline -2bx^2-2b^2x, \text{ first Rem.} & \end{array}$$

Suppressing the factor $-2bx$, we have $x + b$.

$$\begin{array}{l|l} x^2 + 2bx + b^2 & x + b \\ \underline{x^2 + bx \qquad} & \overline{x + b} \\ \quad bx + b^2 & \\ \quad bx + b^2. & \end{array}$$

Whence $x + b$ is the greatest common divisor.

Ex. 3. Required the greatest common divisor of

$$4a^3 - 2a^2 - 3a + 1 \text{ and } 3a^2 - 2a - 1.$$

Ans. $a - 1$.

Ex. 4. Find the greatest common divisor of

$$x^2 - 1 \text{ and } ax + 1.$$

Ex. 5. Find the greatest common divisor of

$$x^4 - a^4 \text{ and } x^3 - a^3.$$

Ex. 6. Find the greatest common divisor of

$$a^2 - 3ab + 2b^2 \text{ and } a^2 - ab - 2b^2.$$

Ex. 7. Find the greatest common divisor of

$$a^4 - x^4 \text{ and } a^3 - a^2x - ax^2 + x^3.$$

Ex. 8. Find the greatest common divisor of

$$a^3 - a^2b + 3ab^2 - 3b^3 \text{ and } a^2 - 5ab + 4b^2.$$

THEORY OF PERMUTATIONS AND COMBINATIONS.

(252.) The different orders in which quantities may be arranged are called their *Permutations*. Thus,

the permutations of the three letters a, b, c, taken *all together*, are . . . .
$$\left\{\begin{array}{l} a,\ b,\ c, \\ a,\ c,\ b, \\ b,\ a,\ c, \\ b,\ c,\ a, \\ c,\ a,\ b, \\ c,\ b,\ a. \end{array}\right.$$

The permutations of the same letters taken *two and two*, are . . . .
$$\left\{\begin{array}{l} a,\ b, \\ a,\ c, \\ b,\ a, \\ b,\ c, \\ c,\ a, \\ c,\ b. \end{array}\right.$$

The permutations of the same letters taken singly, or *one by one*, are . . . . $\left\{\begin{array}{l} a, \\ b, \\ c. \end{array}\right.$

(253.) *To find the number of permutations of* n *letters, taken* m *and* m *together.*

Let $a, b, c, d \ldots\ldots k$, be the n letters.

The number of permutations of n letters taken singly, or one by one, is evidently equal to the number of letters, or to n.

The number of permutations of n letters taken two and two is $n(n-1)$. For if we reserve one of the letters, as a, there will remain $n-1$ letters.

$$b, c, d \ldots\ldots k.$$

Writing a before each of these letters, we shall have

$$ab, ac, ad \ldots\ldots ak;$$

that is, we obtain $n-1$ permutations of the n letters taken two and two, in which *a stands first.* Proceeding in the same manner with b, we shall find $n-1$ permutations of the n letters taken two and two, in which *b stands first;* and so for each of the n letters. Hence the whole number of permutations will be

$$n(n-1).$$

The number of permutations of n letters taken three and three together is

$$n(n-1)(n-2).$$

For if we reserve one of the letters, as a, there will remain $n-1$ letters. Now we have found the number of permutations of n letters taken two and two to be $n(n-1)$. Hence the permutations of $n-1$ letters taken two and two must be

$$(n-1)(n-2).$$

Writing a before each of these permutations, we shall have $(n-1)(n-2)$ permutations of the n letters taken three and three, in which *a stands first.* Proceeding in the same manner with b, we shall find $(n-1)(n-2)$ permutations of the n letters taken three and three, in which *b stands first;* and so for each of the n letters. Hence the whole number of permutations will be

$$n(n-1)(n-2).$$

In like manner, we can prove that the number of permutations of n letters taken four and four is

$$n(n-1)\,(n-2)\,(n-3).$$

When the letters are taken *two and two*, the last factor in the corresponding formula is $n-1$. When the letters are taken *three and three*, the last factor is $n-2$. When the letters are taken *four and four*, the last factor is $n-3$.

Hence, when the letters are taken m and m together, the last factor will be $n-(m-1)$ or $n-m+1$; and the number of permutations of n letters taken m and m together will be

$$n(n-1)\,(n-2)\,(n-3)\,.\,.\,.\,.\,.\,(n-m+1).$$

EXAMPLES.

Ex. 1. Required the number of permutations of the 8 letters a, b, c, d, e, f, g, h, taken 5 and 5 together.

Here $\qquad n=8,\ m=5,\ n-m+1=4,$

and the above formula becomes

$$8.7.6.5.4=6720, \text{ Ans.}$$

Ex. 2. Required the number of permutations of the 26 letters of the alphabet, taken 4 and 4 together.

Ex. 3. Required the number of permutations of 12 letters, taken 6 and 6 together.

(254.) If we suppose that *each permutation comprehends all the* n *letters;* that is, if $m=n$, the preceding formula becomes

$$n(n-1)\,(n-2)\,.\,.\,.\,.\,.\,2\times1;$$

or, inverting the order of the factors,

$$1.2.3.4\,.\,.\,.\,.\,.\,(n-1)n;$$

which expresses the number of permutations of n letters taken all together.

Ex. 1. Required the number of changes which can be rung upon 8 bells.

According to the preceding formula, we have

$$1.2.3.4.5.6.7.8=40320, \text{ Ans.}$$

Ex. 2. What is the number of permutations which may be formed from the letters composing the word "*virtue?*"

Ex. 3. What is the number of different arrangements which can be made of 12 persons at a dinner-table?

(255.) The *combinations* of any number of quantities signify the different collections which may be formed of these quantities, without regard to the *order* of their arrangement.

Thus, the three letters *a*, *b*, *c*, taken *all together*, form but *one* combination, *abc*.

Taken *two and two*, they form three combinations,

$$ab, ac, bc.$$

To find the number of combinations of n *letters, taken* m *and* m *together.*

The number of combinations of n letters taken separately, or *one by one*, is evidently n.

The number of combinations of n letters taken *two and two*, is $\frac{n(n-1)}{1.2}$

For the number of permutations of n letters taken two and two is $n(n-1)$; and there are two permutations (ab, ba) corresponding to one combination of two letters. Therefore the number of combinations will be found by dividing the number of permutations by 2.

The number of combinations of n letters taken *three and three* together, is $\frac{n(n-1)(n-2)}{1.2.3}$.

For the number of permutations of n letters taken three and three, is $n(n-1)(n-2)$; and there are 1.2.3 permutations for one combination of three letters. Therefore the number of combinations will be found by dividing the number of permutations by 1.2.3.

In the same manner, we find the number of combinations of n letters, taken m and m together,

$$\frac{n(n-1)(n-2)\ldots\ldots(n-m+1)}{1.2.3\quad\ldots\ldots\quad m}.$$

Ex. 1. Required the number of combinations of six letters taken three and three together.

Here $n=6$, $m=3$, $n-m+1=4$,

and the formula becomes

$$\frac{6.5.4}{1.2.3}=20.$$

Ex. 2. Required the number of combinations of 8 letters taken 4 and 4.

Ex. 3. Required the number of combinations of 10 letters taken 6 and 6.

The following table, which is computed by the preceding formula, shows the number of combinations of 1, 2, 3, 4, &c., letters taken singly, or two and two, three and three, &c. An important application of this table will be seen in the next Section.

Lett'rs.	Singly.	2 and 2.	3 and 3.	4 and 4.	5 and 5.	6 and 6.	7 and 7.	8 and 8.	9 and 9.	10, 10.
1	1									
2	2	1			Number of combinations.					
3	3	3	1							
4	4	6	4	1						
5	5	10	10	5	1					
6	6	15	20	15	6	1				
7	7	21	35	35	21	7	1			
8	8	28	56	70	56	28	8	1		
9	9	36	84	126	126	84	36	9	1	
10	10	45	120	210	252	210	120	45	10	1

SECTION XVI.

INVOLUTION OF BINOMIALS.

(256.) WE have shown, in Art. 142, how to obtain any power of a binomial by actual multiplication. We now propose to develop a theorem by which this labor may be greatly abridged.

Taking the binomial $a+b$, its successive powers found by actual multiplication are as follows:

$$\begin{aligned}
(a+b)^1 &= a + b,\\
(a+b)^2 &= a^2 + 2ab + b^2,\\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3,\\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,\\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5,\\
(a+b)^6 &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.
\end{aligned}$$

(257.) The powers of $a-b$, found in the same manner, are as follows:

$$\begin{aligned}
(a-b)^1 &= a - b,\\
(a-b)^2 &= a^2 - 2ab + b^2,\\
(a-b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3,\\
(a-b)^4 &= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4,\\
(a-b)^5 &= a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5,\\
(a-b)^6 &= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6.
\end{aligned}$$

On comparing the powers of $a+b$ with those of $a-b$, we perceive that they only differ in *the signs of certain terms.* In the powers of $a+b$, all the terms are positive. In the powers of $a-b$, the terms containing the odd powers of b have the sign $-$, while the even powers retain the sign $+$. The reason of this is obvious; for since $-b$ is the only negative term of the root, the terms of the power can only be rendered negative by b. A term which contains the factor $-b$ an even number of times, will therefore be positive; if it contain it an odd number of times, it must be negative. Hence it appears that it is only necessary to seek for a method of obtaining the powers of $a+b$; for these will become the powers of $a-b$, by simply changing the signs of the alternate terms.

(258.) If we consider the exponents of the preceding powers, we shall find that they follow a very simple law. Thus,

In the square, the exponents . .	of a are	2, 1, 0,
	of b are	0, 1, 2.
In the cube, the exponents . .	of a are	3, 2, 1, 0,
	of b are	0, 1, 2, 3.
In the fourth power, the exponents	of a are	4, 3, 2, 1, 0,
	of b are	0, 1, 2, 3, 4.
&c.,	&c.,	&c.

In the first term of each power, a is raised to the required power of the binomial; and in the following terms the exponents of a continually decrease by unity to 0; while the exponents of b increase by unity from 0 up to the required power of the binomial. The sum of the exponents of a and b in any term is equal to the exponent of the power required. Thus, in the second power, the sum of the exponents of a and b in each term is 2; in the third power, it is 3; in the fourth power, 4, &c.

We hence infer that for the seventh power the terms, without the coefficients, are

$$a^7,\ a^6b,\ a^5b^2,\ a^4b^3,\ a^3b^4,\ a^2b^5,\ ab^6,\ b^7;$$

and for the nth power

$$a^n,\ a^{n-1}b,\ a^{n-2}b^2,\ a^{n-3}b^3 \ldots\ldots a^2b^{n-2},\ ab^{n-1},\ b^n.$$

(259.) It remains to determine the coefficients which belong to these terms; and in order to discover the law of their formation, let us take the coefficients by themselves.

The coefficients of the 1st power are	1	1					
The coefficients of the 2d power are	1	2	1				
The coefficients of the 3d power are	1	3	3	1			
The coefficients of the 4th power are	1	4	6	4	1		
The coefficients of the 5th power are	1	5	10	10	5	1	
The coefficients of the 6th power are	1	6	15	20	15	6	1

The numbers in this table are identical with those in the table of combinations on page 241. For example, the coefficients of the fifth power denote the number of combinations of five letters taken one and one, two and two, &c.; the coefficients of the sixth power denote the number of combinations of six letters taken one and one, two and two, &c. The reason of this will appear if we observe the law of the product of several binomial factors, $x+a$, $x+b$, $x+c$, $x+d$, &c.

Multiply $x+a$

by $\underline{x+b}$,

we obtain $x^2+(a+b)x+ab=$ 1st product.

Multiply by $\underline{x+c}$,

we obtain $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc=$ 2d product.

Multiply by $\underline{x+d}$,

we obtain $x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2+$
$+(abc+abd+acd+bcd)x+abcd=$ 3d product.

We observe that in each of these products *the coefficient of x in the first term is unity; the coefficient of the second term is the sum of the second terms of the binomial factors; the coefficient of the third term is the sum of all their products two and two; the coefficient of the fourth term is the sum of all their products three and three,* &c.

It is easily seen that if we multiply the last product by a new factor, $x+e$, the same law of the coefficients will be preserved.

If now, in the preceding binomial factors, we suppose a, b, c, d, &c., to be all equal to each other, the product

$$(x+a)\,(x+b)\,(x+c)\,(x+d)\,\ldots\ldots$$

becomes

$$(x+a)^n.$$

The coefficient of the second term of the product, or $a+b+c+d$ ....., becomes $a+a+a+a$ .....; that is, a taken as many times as there are letters a, b, c, d, and is consequently equal to na.

The coefficient of the third term, or $ab+ac$, &c., reduces to $a^2+a^2+a^2$ ....., or a^2 repeated as many times as there are different combinations of n letters taken two and two; that is, $\frac{n(n-1)}{1.2}a^2$.

The coefficient of the fourth term reduces to a^3 repeated as many times as there are different combinations of n letters taken three and three, that is, $\frac{n(n-1)(n-2)}{1.2.3}a^3$, and so on.

Thus, we find that the nth power of $x+a$ may be expressed as follows:

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1.2}a^2x^{n-2} + \frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-3} + \ldots$$
$$+ na^{n-1}x + a^n,$$

which is called the Binomial Theorem, and is generally ascribed to Sir Isaac Newton. So important was it regarded, that it was engraved on his monument in Westminster Abbey as one of his greatest discoveries.

On comparing the different terms of this development, we perceive that any coefficient may be derived from the preceding one by the following rule: *If the coefficient of any term be multiplied by the index of* x *in that term, and divided by the index of* a *increased by one, it will give the coefficient of the succeeding term.*

Thus, the fifth power of $x+a$ is

$$x^5 + 5ax^4 + 10a^2x^3 + 10a^3x^2 + 5a^4x + a^5.$$

If the coefficient 5 of the second term be multiplied by 4, the index of x in that term, and divided by 2, which is the index of a increased by one, we obtain 10, the coefficient of the third term.

So, also, if 10, the coefficient of the fourth term, be multiplied by 2, the index of x, and divided by 4, the index of a increased by one, we obtain 5, the coefficient of the fifth term; and so of the others.

The coefficients of the sixth power will also be found as follows:

$$1,\ 6,\ \frac{6\times 5}{2},\ \frac{15\times 4}{3},\ \frac{20\times 3}{4},\ \frac{15\times 2}{5},\ \frac{6\times 1}{6};$$

that is, 1, 6, 15, 20, 15, 6, 1.

The coefficients of the seventh power will be

$$1,\ 7,\ \frac{7\times 6}{2},\ \frac{21\times 5}{3},\ \frac{35\times 4}{4},\ \frac{35\times 3}{5},\ \frac{21\times 2}{6},\ \frac{7\times 1}{7};$$

that is, 1, 7, 21, 35, 35, 21, 7, 1.

Therefore, the seventh power of $x+a$ is

$$x^7 + 7ax^6 + 21a^2x^5 + 35a^3x^4 + 35a^4x^3 + 21a^5x^2 + 7a^6x + a^7.$$

It is sometimes preferable to retain the factors of the coefficients distinct from each other, as follows:

$$(x+a)^7=x^7+\frac{7}{1}ax^6+\frac{7.6}{1.2}a^2x^5+\frac{7.6.5}{1.2.3}a^3x^4+\frac{7.6.5.4}{1.2.3.4}a^4x^3+\frac{7.6.5.4.3}{1.2.3.4.5}a^5x^2+$$
$$+\frac{7.6.5.4.3.2}{1.2.3.4.5.6}a^6x+\frac{7.6.5.4.3.2.1}{1.2.3.4.5.6.7}a^7.$$

The factor 1 is retained for the sake of symmetry, and to exhibit more clearly the law of the coefficients.

(260.) The theorem thus developed is expressed in the following

RULE.

In any power of a binomial x + a, *the index of* x *begins in the first term with the index of the power, and in the following terms continually decreases by one. The index of* a *commences with one in the second term of the power, and continually increases by one.*

The coefficient of the first term is one; that of the second is the index of the power; and if the coefficient of any term be multiplied by the index of x *in that term, and divided by the index of* a *increased by one, it will give the coefficient of the succeeding term.*

(261.) The *number of terms* in the power is always greater by unity than the index of the power. Thus, the number of terms in $(a+b)^4$ is 4 + 1 or 5; in $(a+b)^6$ is 6 + 1 or 7.

Also, if we examine the table in Art. 259, it will be perceived that after we pass the middle term, the same coefficients are repeated in the inverse order. Thus, the coefficients of

$(a+b)^5$ are 1, 5, 10, 10, 5, 1,
of $(a+b)^6$ are 1, 6, 15, 20, 15, 6, 1.

Hence it is only necessary to compute the coefficients for *half* the terms; we then repeat the same numbers in the inverse order.

(262.) The *sum of the coefficients* for each power is equal to the number 2 raised to the same power. For, let $x=1$ and $a=1$, then each term without the coefficients reduces to unity, and the value of the power is simply the sum of the coefficients. Also, in this case, $(x+a)^n$ becomes 2^n. Thus, the coefficients of the

first power are $1+1=2=2^1$,
second power are $1+2+1=4=2^2$,
third power are $1+3+3+1=8=2^3$,
fourth power are $1+4+6+4+1=16=2^4$.
&c., &c., &c.

EXAMPLES.

Ex. 1. Raise $x + a$ to the 9th power.

The terms without the coefficients are

$$x^9,\ ax^8,\ a^2x^7,\ a^3x^6,\ a^4x^5,\ a^5x^4,\ a^6x^3,\ a^7x^2,\ a^8x,\ a^9.$$

And the coefficients are

$$1,\ 9,\ \frac{9\times 8}{2},\ \frac{36\times 7}{3},\ \frac{84\times 6}{4},\ \frac{126\times 5}{5},\ \frac{126\times 4}{6},\ \frac{84\times 3}{7},\ \frac{36\times 2}{8},\ \frac{9\times 1}{9},$$

that is, 1, 9, 36, 84, 126, 126, 84, 36, 9, 1.

Prefixing the coefficients, we obtain

$$(x+a)^9=x^9+9ax^8+36a^2x^7+84a^3x^6+126a^4x^5+126a^5x^4+84a^6x^3+$$
$$+36a^7x^2+9a^8x+a^9.$$

It should be remembered that, according to Art. 261, it is only necessary to compute the coefficients of *half* the terms independently.

Ex. 2. What is the 6th power of $x - a$?

(263.) If the terms of the given binomial are affected with coefficients or exponents, they must be raised to the required powers, according to the principles already established for the involution of monomials.

Ex. 3. Raise $2x + 5a^2$ to the fourth power.

For convenience, let us substitute b for $2x$, and c for $5a^2$.

Then $(b + c)^4 = b^4 + 4b^3c + 6b^2c^2 + 4bc^3 + c^4.$

Restoring the values of b and c,

The first term will be $(2x)^4 = 16x^4$.
The second term will be $4(2x)^3 . 5a^2 = 4.8.5x^3a^2$.
The third term will be $6(2x)^2 . (5a^2)^2 = 6.4.25x^2a^4$.
The fourth term will be $4(2x) . (5a^2)^3 = 4.2.125xa^6$.
The fifth term will be $(5a^2)^4 = 625a^8$.

Therefore, $(2x+5a^2)^4=16x^4+160x^3a^2+600x^2a^4+1000xa^6+625a^8$.

Ex. 4. What is the fourth power of $2x^3 + 5y^2$?

Ex. 5. What is the seventh power of $2a + 3b$?

Ex. 6. What is the sixth power of $a^3 + 3ab$?

Ex. 7. What is the fourth power of $5c^2 - 2yz$?

(264.) By means of the Binomial Theorem we can raise any polynomial to any power.

For example, let it be required to raise $a+b+c$ to the third power.

For convenience, we put $b+c=m$; we then obtain

$$(a+b+c)^3=(a+m)^3=a^3+3a^2m+3am^2+m^3.$$

Substituting for m its equal, $b+c$, we obtain

$$(a+b+c)^3=a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3.$$

It only remains to develop the powers of the binomial $b+c$, and to perform the multiplications which are indicated. We thus obtain

$$\begin{aligned}(a+b+c)^3=a^3+3a^2b&+3ab^2+b^3,\\ +3a^2c&+6abc+3b^2c,\\ &+3ac^2+3bc^2,\\ &\qquad+c^3.\end{aligned}$$

Ex. 2. Raise $x+a+b$ to the fifth power.

(265.) When one of the terms of a binomial is unity, the powers assume a simpler form, since every power of 1 is 1.

Thus, the fourth power of $a+b$, which is

$$a^4+4a^3b+6a^2b^2+4ab^3+b^4,$$

when we make $a=1$, becomes

$$1+4b+6b^2+4b^3+b^4.$$

So, also, $(1+a)^n=1+\frac{n}{1}a+\frac{n(n-1)}{1.2}a^2+\frac{n(n-1)(n-2)}{1.2.3}a^3+$, &c.

Every binomial of the form $(x+a)^n$ may be reduced to the form of $x^n(1+b)^n$. For

$$x+a=x\left(1+\frac{a}{x}\right).$$

Therefore, $(x+a)^n=x^n\left(1+\frac{a}{x}\right)^n$; or, putting $b=\frac{a}{x}$,

$$=x^n(1+b)^n,$$

$$=x^n\left[1+n.\frac{a}{x}+\frac{n(n-1)}{1.2}.\frac{a^2}{x^2}+\frac{n(n-1)(n-2)}{1.2.3}\frac{a^3}{x^3}+,\&c.\right].$$

This expression for the value of $(x+a)^n$ is equivalent to that on page 245, as may be easily shown by multiplying x^n into each term within the parenthesis. For some purposes this is regarded as the simplest form.

(266.) In the development of the binomial $(x+a)^n$, we have hitherto supposed n to be a *positive integer*. The Binomial Theorem is, however, applicable, whatever be the nature of the quantity n, whether it be *positive or negative, integral or fractional*. When n is a positive integer, the series consists of $n+1$ terms. In every other case, the series never terminates; that is, the development of $(x+a)^n$ furnishes an *infinite series*.

Ex. 1. It is required to convert $\frac{1}{a+b}$ or $(a+b)^{-1}$ into an infinite series.

According to Art. 260, the terms without the coefficients are

$$a^{-1},\ a^{-2}b,\ a^{-3}b^2,\ a^{-4}b^3,\ a^{-5}b^4,\ a^{-6}b^5,\ a^{-7}b^6,\ \text{\&c.}$$

The coefficient of the first term is 1.

The coefficient of the 2d term is -1, the index of the power.

The coefficient of the third term is $\frac{-1\times-2}{2}=+1$.

The coefficient of the fourth term is $\frac{+1\times-3}{3}=-1$.

The coefficient of the fifth term is $\frac{-1\times-4}{4}=+1$.

The coefficient of the sixth term is $\frac{+1\times-5}{5}=-1$.

We thus obtain

$$\frac{1}{a+b}=(a+b)^{-1}=a^{-1}-a^{-2}b+a^{-3}b^2-a^{-4}b^3+a^{-5}b^4-a^{-6}b^5+,\text{\&c.},$$

where the law of the series is obvious; the coefficients are all unity, and the signs are alternately positive and negative.

We might have obtained the same result by the ordinary method of division. The operation is as follows:

$$\begin{array}{l|l}
1 & a+b \\ \cline{2-2}
1+\frac{b}{a} & \frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+,\text{\&c.},=\text{the quotient.} \\ \cline{1-1}
-\frac{b}{a} & \\
-\frac{b}{a}-\frac{b^2}{a^2} & \\ \cline{1-1}
+\frac{b^2}{a^2} & \\
+\frac{b^2}{a^2}+\frac{b^3}{a^3} & \\ \cline{1-1}
-\frac{b^3}{a^3}. &
\end{array}$$

Hence,

$\frac{1}{a+b}=\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}$, &c.; which may be written

$a^{-1}-a^{-2}b+a^{-3}b^2-a^{-4}b^3+$, &c., the same as before found;

and it is obvious, from inspecting the operation of division, that the series will never terminate.

Ex. 2. It is required to convert $\frac{1}{(a+b)^2}$ or $(a+b)^{-2}$ into an infinite series

Ans. $\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}-$, &c.;

or, $a^{-2}-2a^{-3}b+3a^{-4}b^2-4a^{-5}b^3+5a^{-6}b^4$, &c.

Here the coefficients increase regularly by 1, and the signs are alternately positive and negative. We might have obtained the same result by division, as in the former example

Ex. 3. Expand into a series $\frac{1}{a-b}$ or $(a-b)^{-1}$.

Ex. 4. Expand into a series $\frac{1}{(a-b)^2}$ or $(a-b)^{-2}$.

Ex. 5. Expand into a series $(a+b)^{-3}$.

Ex. 6. Expand into a series $(a-b)^{-4}$.

(267.) We have now considered the powers of a binomial when the exponent is an integer, either positive or negative. It remains to consider the case when the exponent is a *fraction.*

EXAMPLES.

Ex. 1. Expand $\sqrt{a+b}$ or $(a+b)^{\frac{1}{2}}$ into an infinite series.
The terms without the coefficients are

$$a^{\frac{1}{2}},\ a^{-\frac{1}{2}}b,\ a^{-\frac{3}{2}}b^2,\ a^{-\frac{5}{2}}b^3,\ a^{-\frac{7}{2}}b^4, \text{ \&c.}$$

The exponents of *a* decrease by unity, while those of *b* increase by unity.

The coefficient of the first term is $1.$
The coefficient of the second term is $+\frac{1}{2}.$
The coefficient of the third term is $\frac{\frac{1}{2}\times-\frac{1}{2}}{2}=-\frac{1}{2.4}.$

The coefficient of the fourth term is $\dfrac{-\frac{1}{2.4}\times-\frac{3}{2}}{3}=+\dfrac{1.3}{2.4.6}$.

The coefficient of the fifth term is $\dfrac{\frac{1.3}{2.4.6}\times-\frac{5}{2}}{4}=-\dfrac{1.3.5}{2.4.6.8}$.

The series then is

$$(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}b-\frac{1}{2.4}a^{-\frac{3}{2}}b^2+\frac{1.3}{2.4.6}a^{-\frac{5}{2}}b^3-\frac{1.3.5}{2.4.6.8}a^{-\frac{7}{2}}b^4+,\&c.$$

The factors which form the coefficients are kept distinct, in order to show more clearly the *law* of the series. The factors in the numerator compose the series of odd numbers, 1, 3, 5, 7, &c.; while the denominator contains the even numbers, 2, 4, 6, 8, 10, &c.

The above series expresses the square root of $a+b$. We should obtain the same result if we extracted the square root by the usual method. See Art. 286.

Ex. 2. It is required to convert $(a^2+x)^{\frac{1}{2}}$ into an infinite series.

$$\text{Ans. } a+\frac{x}{2a}-\frac{x^2}{2.4a^3}+\frac{3x^3}{2.4.6a^5}-\frac{3.5x^4}{2.4.6.8a^7}+\frac{3.5.7x^5}{2.4.6.8.10a^9}-,\&c.;$$

$$\text{or, } a+\frac{a^{-1}x}{2}-\frac{a^{-3}x^2}{2.4}+\frac{3a^{-5}x^3}{2.4.6}-\frac{3.5.a^{-7}x}{2.4.6.8}+\frac{3.5.7a^{-9}x^5}{2.4.6.8.10}-,\&c.,$$

where the law of the series is evident.

Ex. 3. It is required to convert $(a-x)^{\frac{1}{2}}$ into an infinite series.

Ex. 4. It is required to convert $(a+b)^{\frac{1}{3}}$ into an infinite series.

$$\text{Ans. } a^{\frac{1}{3}}\left\{1+\frac{b}{3a}-\frac{2b^2}{3.6a^2}+\frac{2.5b^3}{3.6.9a^3}-\frac{2.5.8b^4}{3.6.9.12a^4}+,\&c.\right\}.$$

Ex. 5. Expand $(a-b)^{\frac{1}{4}}$ into an infinite series.

$$\text{Ans. } a^{\frac{1}{4}}\left\{1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}-,\&c.\right\}.$$

Ex. 6. Expand $(a+x)^{\frac{2}{3}}$ into an infinite series.

Ex. 7. Expand $(1-x)^{\frac{1}{5}}$ into an infinite series.

Ex. 8. Expand $(a^2+b^2)^{\frac{1}{2}}$ into an infinite series.

Ex. 9. Expand $(a^3-b^3)^{\frac{1}{3}}$ into an infinite series.

(268.) The binomial theorem is applicable to cases in which the value of the exponent n is a *negative fraction*.

EXAMPLES.

Ex. 1. Expand into a series $\frac{1}{(a+b)^{\frac{1}{2}}}$ or $(a+b)^{-\frac{1}{2}}$.

The terms without the coefficients are

$$a^{-\frac{1}{2}},\ a^{-\frac{3}{2}}b,\ a^{-\frac{5}{2}}b^2,\ a^{-\frac{7}{2}}b^3,\ a^{-\frac{9}{2}}b^4,\ a^{-\frac{11}{2}}b^5,\ \&c.$$

The coefficient of the first term is 1.

The coefficient of the second term is $-\frac{1}{2}$.

The coefficient of the third term is $\frac{-\frac{1}{2}\times-\frac{3}{2}}{2}=+\frac{1.3}{2.4}$.

The coefficient of the fourth term is $\frac{\frac{1.3}{2.4}\times-\frac{5}{2}}{3}=-\frac{1.3.5}{2.4.6}$.

The coefficient of the fifth term is $\frac{-\frac{1.3.5}{2.4.6}\times-\frac{7}{2}}{4}=+\frac{1.3.5.7}{2.4.6.8}$.

Hence we obtain

$$(a+b)^{-\frac{1}{2}}=a^{-\frac{1}{2}}-\frac{1}{2}a^{-\frac{3}{2}}b+\frac{1.3}{2.4}a^{-\frac{5}{2}}b^2-\frac{1.3.5}{2.4.6}a^{-\frac{7}{2}}b^3+\frac{1.3.5.7}{2.4.6.8}a^{-\frac{9}{2}}b^4-$$
$$-,\ \&c.$$

Ex. 2. Expand into an infinite series $(a+x)^{-\frac{1}{3}}$.

Ans. $a^{-\frac{1}{3}}-\frac{1}{3}a^{-\frac{4}{3}}x+\frac{1.4}{3.6}a^{-\frac{7}{3}}x^2-\frac{1.4.7}{3.6.9}a^{-\frac{10}{3}}x^3+\frac{1.4.7.10}{3.6.9.12}a^{-\frac{13}{3}}x^4-$, &c.

Ex. 3. Expand $(1+x)^{-\frac{1}{5}}$ into an infinite series.

Ans. $1-\frac{x}{5}+\frac{6x^2}{5.10}-\frac{6.11.x^3}{5.10.15}+\frac{6.11.16x^4}{5.10.15.20}-$, &c.

Ex. 4. Expand $(a^2-x)^{-\frac{1}{2}}$ into an infinite series.

Ex. 5. Expand $\frac{m}{\sqrt{b^2+c^4}}$ into an infinite series.

(269.) The binomial theorem may be employed to determine the roots of surd numbers.

EXAMPLES.

Ex. 1. It is required to find the square root of 2.

The development of $(a+b)^{\frac{1}{2}}$ has been given in Example 1, page 251. If we make $a=1$ and $b=1$, then $(a+b)^{\frac{1}{2}}$ becomes $(1+1)^{\frac{1}{2}}$ or $\sqrt{2}$; and the terms of the development become

$$1+\frac{1}{2}-\frac{1}{2.4}+\frac{1.3}{2.4.6}-\frac{1.3.5}{2.4.6.8}+\frac{1.3.5.7}{2.4.6.8.10}-, \&c.,$$

which therefore expresses the square root of 2. The sum of this series is 1.41421. As, however, the series does not converge rapidly, it would require a large number of terms to give the root with tolerable accuracy. The following example affords a better illustration of the *utility* of the method.

Ex. 2. Required the square root of 101.

$$101=100\left(1+\frac{1}{100}\right). \quad \text{Therefore } \sqrt{101}=10\left(1+\frac{1}{100}\right)^{\frac{1}{2}}.$$

Put $a=1$ and $b=\frac{1}{100}$ in the development of $(a+b)^{\frac{1}{2}}$ on page 251, and we shall have

$$\sqrt{101}=10\left(1+\frac{1}{2.100}-\frac{1}{2.4.100^2}+\frac{1.3}{2.4.6.100^3}-\frac{1.3.5}{2.4.6.8.100^4}+, \&c.\right)$$

This series converges so rapidly that the first two terms give a result correct to three decimal places; and five terms give a result correct to ten decimal places.

Thus, the value of the first term is	1.00000000000
The value of the second term is	+ .00500000000
The value of the third term is	— .00001250000
The value of the fourth term is	+ .00000006250
The value of the fifth term is	— .00000000039
Their sum is	1.00498756211

And multiplying by 10, we have

$$\sqrt{101}=10.0498756211.$$

Ex. 3. It is required to convert $\sqrt[3]{9}$, or its equal $(8+1)^{\frac{1}{3}}$, into an infinite series, and find its value.

$$\text{Ans. } 2+\frac{1}{3.2^2}-\frac{1}{3.6.2^4}+\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}+\frac{5.8.11}{3.6.9.12.15.2^{13}}-, \&c., \\ =2.08008.$$

Ex. 4. It is required to extract the cube root of 31.

$$\sqrt[3]{31}=\sqrt[3]{27+4}=\sqrt[3]{27}\left(1+\frac{4}{27}\right)^{\frac{1}{3}},$$

$$=3\left\{1+\frac{4}{3.27}-\frac{2.4^2}{3.6.27^2}+\frac{2.5.4^3}{3.6.9.27^3}-\frac{2.5.8.4^4}{3.6.9.12.27^4}+,\ \&c.\right\},$$

$$=3.14138.$$

SECTION XVII.

EVOLUTION OF POLYNOMIALS.

(270.) *Method of extracting the square root.*

In order to discover a rule for extracting the square root, let us consider the square of $a+b$, which is $a^2+2ab+b^2$. If we write the terms of the square in such a manner that the powers of one of the letters, as a, may go on continually decreasing, the first term will be the square of the first term of the root; and since in the present case the first term of the square is a^2, the first term of the root must be a.

Having found the first term of the root, we must consider the rest of the square, namely, $2ab+b^2$, to see how we can derive from it the second term of the root. Now this remainder, $2ab+b^2$, may be put under the form $(2a+b)b$; whence it appears that we shall find the second term of the root if we divide the remainder by $2a+b$. The first part of this divisor, $2a$, is double of the first term already determined; the second part, b, is yet unknown, and it is necessary at present to leave its place empty. Nevertheless, we may commence the division, employing only the term $2a$; but as soon as the quotient is found, which, in the present case, is b, we must put it in the vacant place, and thus render the divisor complete.

The whole process, therefore, may be represented as follows:

$$\begin{array}{r|l} a^2+2ab+b^2 & \underline{a+b} = \text{the root.} \\ a^2 \qquad\qquad & \\ \cline{1-1} 2ab+b^2 & 2a+b = \text{the divisor.} \\ 2ab+b^2 & \end{array}$$

Hence we derive the following

RULE FOR EXTRACTING THE SQUARE ROOT OF A POLYNOMIAL.

Arrange the polynomial according to the powers of some one letter; take the root of the first term for the first term of the required root, and subtract its square from the given quantity.

Divide the first term of the remainder by double the root already found, and annex the result both to the root and the divisor. Multiply the divisor thus increased by the last term of the root, and subtract the product from the last remainder. Proceed in the same manner to find the additional terms of the root.

Ex. 1. Required the square root of $a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4$.

$$\begin{array}{l|l}
a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4 & \underline{a^2} - ax + x^2 = \text{the root.} \\
a^4 & \\
\hline
\quad -2a^3x + 3a^2x^2 & 2a^2 - ax = \text{the first divisor.} \\
\quad -2a^3x + a^2x^2 & \\
\hline
\qquad\qquad 2a^2x^2 - 2ax^3 + x^4 & 2a^2 - 2ax + x^2 = \text{second divisor.} \\
\qquad\qquad \underline{2a^2x^2 - 2ax^3 + x^4} &
\end{array}$$

For verification, multiply this root by itself thus:

$$\begin{array}{llll}
a^2 - ax & + x^2 & & \\
a^2 - ax & + x^2 & & \\
\hline
a^4 - a^3x & + a^2x^2 & & \\
\quad - a^3x & + a^2x^2 & - ax^3 & \\
 & + a^2x^2 & - ax^3 & + x^4 \\
\hline
a^4 - 2a^3x & + 3a^2x^2 & - 2ax^3 & + x^4,
\end{array}$$

which is the original polynomial.

Ex. 2. Required the square root of $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$.

Ex. 3. Required the square root of $10x^4 - 10x^3 - 12x^5 + 5x^2 + 9x^6 - 2x + 1$.

Ex. 4. Required the square root of $4x^4 + 8ax^3 + 4a^2x^2 + 16b^2x^2 + 16ab^2x + 16b^4$.

Ans. $2x^2 + 2ax + 4b^2$.

Ex. 5. Extract the square root of $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$.

Ex. 6. Extract the square root of $a^4 - 4a^3b + 8ab^3 + 4b^4$.

(271.) *To extract the cube root of a polynomial.*

We already know that the cube of $a+b$ is $a^3 + 3a^2b + 3ab^2 + b^3$.

If, then, the *cube* were given, and we were required to find its *root*, it might be done by the following method:

When the cube is arranged according to the powers of one letter, we at once know from the first term a^3, that a must be one term of the root. If, then, we subtract its cube from the proposed polynomial, we obtain the remainder $3a^2b + 3ab^2 + b^3$, which must furnish the second term of the root.

Now this remainder may be put under the form

$$(3a^2 + 3ab + b^2) \times b;$$

whence it appears that we shall find the second term of the root, if we divide the remainder by $3a^2 + 3ab + b^2$. But as this second term is supposed to be unknown, the divisor cannot be completed. Nevertheless, we know the first term $3a^2$, that is, thrice the square of the first term already found, and by means of this we can find the other part b, and then complete the divisor before we perform the division. For this purpose, we must add to $3a^2$ thrice the product of the two terms, or $3ab$, and the square of the second term of the root, or b^2. Hence we derive the following

RULE FOR EXTRACTING THE CUBE ROOT OF A POLYNOMIAL.

(272.) *Having arranged the polynomial according to the powers of some one letter, take the cube root of the first term, and subtract the cube from the given quantity.*

Divide the first term of the remainder by three times the square of the root already found, the quotient will be the second term of the root.

Complete the divisor by adding to it three times the product of the two terms of the root, and the square of the second term.

Multiply the divisor thus increased by the last term of the root, and subtract the product from the last remainder. Proceed in the same manner to find the additional terms of the root.

Ex. 1. Extract the cube root of $a^3 + 12a^2 + 48a + 64$.

$$\begin{array}{r|l} a^3 + 12a^2 + 48a + 64 & \underline{a + 4} = \text{the root.} \\ a^3 \qquad\qquad\qquad\quad & \\ \hline 12a^2 + 48a + 64 & 3a^2 + 12a + 16 = \text{the divisor.} \\ 12a^2 + 48a + 64 & \\ \hline \cdot\,\cdot \qquad \cdot\,\cdot \qquad \cdot\,\cdot & \end{array}$$

Having found the first term of the root a, and subtracted its cube, we divide the first term of the remainder, $12a^2$, by three times the square of a, that is, $3a^2$, and we obtain 4 for the second term of the root. We then complete the divisor by adding to it three times the product of the two terms of the root, which is $12a$, together with the square of the last term, 4, which is 16. Multiplying, then, the complete divisor by 4, and subtracting the product from the last remainder, nothing is left. Hence the required cube root is $a + 4$.

This result may be easily verified by multiplication.

Ex. 2. Extract the cube root of $a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1$.

$$\begin{array}{l|l}
a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1 & a^2 - 2a + 1 = \text{the root.} \\
a^6 & \\
\hline
\quad - 6a^5 + 15a^4 - 20a^3 & 3a^4 - 6a^3 + 4a^2 = \text{the first divisor.} \\
\quad - 6a^5 + 12a^4 - 8a^3 & \\
\hline
\qquad\qquad 3a^4 - 12a^3 + 15a^2 - 6a + 1 & 3a^4 - 12a^3 + 15a^2 - 6a + 1 \\
\qquad\qquad 3a^4 - 12a^3 + 15a^2 - 6a + 1 & \quad = \text{second divisor.} \\
\hline
\qquad\qquad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot &
\end{array}$$

We may dispense with forming the complete divisor according to the rule, if, each time that we find a new term of the root, we raise the entire root to the third power, and subtract the cube from the given polynomial.

Ex. 3. Required the cube root of $x^6 + 6x^5 - 40x^3 + 96x - 64$.

Ex. 4. Required the cube root of $18x^4 + 36x^2 + 24x + 8 + 32x^3 + x^6 + 6x^5$.

Ex. 5. Required the cube root of $b^6 - 5b^3 + 3b^5 - 1 + 3b$.

(273.) *To extract any root of a polynomial.*

We already know that the nth power of $a + b$ is $a^n + na^{n-1}b +$ other terms. The first term of the root is, therefore, the nth root of the first term of the polynomial. Also, the second term of the root may be found by dividing the second term of the polynomial by na^{n-1}; that is, the first term of the root raised to the next inferior power, and multiplied by the index of the given power. Hence we deduce the following

RULE FOR EXTRACTING ANY ROOT OF A POLYNOMIAL.

Having arranged the terms according to the powers of one of the letters, take the n*th root of the first term for the first term of the required root.*

Subtract its power from the given quantity, and divide the first term of the remainder, by n *times the* (n — 1) *power of this root; the quotient will be the second term of the root.*

Subtract the n*th power of the terms already found, from the given quantity, and using the same divisor, proceed in like manner to find the remaining terms of the root.*

Ex. 1. Required the fourth root of $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$.

$$\begin{array}{l|l}
16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4 & \underline{2a} - 3x = \text{the root.} \\
16a^4 & \\
\hline
\quad - 96a^3x & 32a^3 = \text{the divisor.} \\
\hline
16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4. &
\end{array}$$

Here we take the fourth root of $16a^4$, which is $2a$, for the first term of the required root; subtract its fourth power, and bring down the first term of the remainder $-96a^3x$. For a divisor, we raise the first term of the root to the third power, and multiply it by 4, making $32a^3$. Dividing, we obtain $-3x$ for the second term of the root. The quantity $2a - 3x$ being raised to the fourth power, is found to be equal to the proposed polynomial.

Ex. 2. Required the fifth root of $32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$.

Ans. $2x - 1$.

Ex. 3. Required the fourth root of $81x^8 - 216x^7 + 336x^5 - 56x^4 - 224x^3 + 64x + 16$.

Ans. $3x^2 - 2x - 2$.

(274.) When the index of the root to be extracted is a multiple of two or more numbers, we may obtain the root required by the successive extraction of simpler roots, Art. 159.

For example, we may obtain the *fourth* root by extracting the square root *twice* successively; for the square root of a^4 is a^2, and the square root of a^2 is a.

The *eighth* root may be obtained by extracting the square root *three times* successively; for the square root of a^8 is a^4, that of a^4 is a^2, and that of a^2 is a.

In the same manner, the *sixteenth* root may be obtained by extracting the square root *four* times successively, and so on.

The *sixth* root may be found by extracting the square root, and afterward the cube root; for the square root of a^6 is a^3, and the cube root of a^3 is a. We may also take, first, the cube root, which gives a^2, and afterward the square root, which gives a, as before. It is, however, best to extract the roots of the lowest degree first, because the operation is less laborious.

In general, *the* m*n*th *root of a number is equal to the* n*th root of the* m*th root of this number.* That is,

$$\sqrt[mn]{a}=\sqrt[n]{\sqrt[m]{a}}.$$

For raising each member of this expression to the nth power, we have

$$\sqrt[m]{a}=\sqrt[m]{a}.$$

Ex. 1. Find the fourth root of $a^4-4a^3b+6a^2b^2-4ab^3+b^4$

Ex. 2. Find the sixth root of $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

Ex. 3. Find the eighth root of $256x^8+1024x^7y+1792x^6y^2+1792x^5y^3+1120x^4y^4+448x^3y^5+112x^2y^6+16xy^7+y^8$.

EXTRACTION OF THE SQUARE ROOT OF A QUANTITY OF THE FORM $a\pm\sqrt{b}$.

(275.) Binomials of this class require particular attention, because they frequently occur in the solution of equations of the fourth degree, such as are treated of in Art. 184. Thus, the equation

$$x^4=6x^2-4,$$

gives us

$$x^2=3\pm\sqrt{5}.$$

Hence, in order to find the value of x, we must extract the square root of the binomial $3\pm\sqrt{5}$.

In order to show that the square root of such an expression may sometimes be extracted, take the binomial

$$2+\sqrt{3},$$

and find its square.

$$(2+\sqrt{3})^2=4+2\sqrt{3}+3=7+2\sqrt{3}.$$

Therefore, the square root of $7+2\sqrt{3}$ is $2+\sqrt{3}$.

The square root of an expression of the form $a\pm\sqrt{b}$ may, therefore, sometimes be extracted, and it is required to determine a general method for this purpose whenever it is practicable.

PROPOSITION I.

The square root of a whole number cannot consist of two parts one of which is rational and the other irrational.

For, if possible, let $\sqrt{a}=x\pm\sqrt{y}$, where x denotes a rational quantity, and $\sqrt{y}$ denotes a *surd.*

By squaring both sides, we obtain $a=x^2\pm 2x\sqrt{y}+y$; whence, by transposition and division,

$$\pm\sqrt{y}=\frac{a-x^2-y}{2x}.$$

The second member of the equation contains only rational quantities, while $\sqrt{y}$ was supposed to be *irrational;* that is, we find an irrational quantity equal to a rational one, which is absurd. Hence the square root of a whole number cannot be partly rational and partly irrational; in other words, the sum or difference of two surds cannot be equal to a whole number.

PROPOSITION II.

In every equation of the form

$$x\pm\sqrt{y}=a\pm\sqrt{b},$$

the rational parts on the opposite sides are equal to each other, and also the irrational parts.

For if x is not equal to a, let it be equal to $a\pm z$.

Then $$a\pm z\pm\sqrt{y}=a\pm\sqrt{b};$$

or $$\pm\sqrt{b}=\pm z\pm\sqrt{y};$$

that is, $\sqrt{b}$ is partly rational and partly irrational, which by the last Proposition is impossible. Therefore, $x=a$, and, consequently, $\sqrt{y}=\sqrt{b}$.

PROPOSITION III.

If $\sqrt{a+\sqrt{b}}$ *is equal to* $x+\sqrt{y}$,

then will $\sqrt{a-\sqrt{b}}$ *be equal to* $x-\sqrt{y}$.

For, by involution, $a+\sqrt{b}=x^2+2x\sqrt{y}+y$.

But by the last Proposition, $a=x^2+y$,

and $\sqrt{b}=2x\sqrt{y}$.

Subtracting, we obtain, $a-\sqrt{b}=x^2-2x\sqrt{y}+y$.

Therefore, by evolution, $\sqrt{a-\sqrt{b}}=x-\sqrt{y}$.

(277.) *To find an expression for the square root of* a ± √ b.

Assume $\sqrt{a+\sqrt{b}}=p+q$ (1),

where p and q may be both radicals, or one rational and the other a radical, but p^2 and q^2 are necessarily *rational*.

Then, by the last Proposition,

$$\sqrt{a-\sqrt{b}}=p-q \quad (2).$$

Multiplying these equations together, we obtain

$$\sqrt{a^2-b}=p^2-q^2 \quad (3), \text{ a rational quantity.}$$

Hence we see that, in order that $\sqrt{a+\sqrt{b}}$ may be expressed by the sum of two radicals, or one rational term and the other a radical, *the expression* a^2-b *must be a perfect square.*

Let, then, a^2-b be a perfect square, and put $\sqrt{a^2-b}=c$; equation (3) will thus become

$$p^2-q^2=c.$$

Squaring equations (1) and (2), we obtain

$$p^2+q^2+2pq=a+\sqrt{b},$$
$$p^2+q^2-2pq=a-\sqrt{b}.$$

Adding these two equations, we obtain

$$p^2+q^2=a.$$

But we have already

$$p^2-q^2=c.$$

Hence

$$2p^2=a+c,$$
$$2q^2=a-c.$$

And, therefore, $$p = \pm\sqrt{\frac{a+c}{2}},$$

$$q = \pm\sqrt{\frac{a-c}{2}}.$$

Therefore,

$$\sqrt{a+\sqrt{b}}, \text{ or } p+q = \pm\left(\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}\right)$$

$$\sqrt{a-\sqrt{b}}, \text{ or } p-q = \pm\left(\sqrt{\frac{a+c}{2}}-\sqrt{\frac{a-c}{2}}\right).$$

(278.) Hence, to extract the square root of a binomial of the form $a \pm \sqrt{b}$, we have the following

RULE.

From the square of the rational part (a^2), *subtract the square of the irrational part* (b); *take the square root of the remainder, and calling that root* c, *the required root will be*

$$\sqrt{\frac{a+c}{2}} \pm \sqrt{\frac{a-c}{2}}.$$

Ex. 1. Required the square root of $4+2\sqrt{3}$.

Here $a=4$, and $\sqrt{b}=2\sqrt{3}$; therefore, $a^2-b=c^2=16-12=4$; or $c=2$. Hence, by the above formula, the required root will be

$$\sqrt{\frac{4+2}{2}}+\sqrt{\frac{4-2}{2}}=\sqrt{3}+1.$$

Verification.

The square of $\sqrt{3}+1$ is $3+2\sqrt{3}+1=4+2\sqrt{3}$.

Ex. 2. Required the square root of $11+6\sqrt{2}$.

Here $a=11$, and $\sqrt{b}=6\sqrt{2}$; therefore, $b=36\times2=72$; and $a^2-b=49=c^2$. Hence $c=7$, and we find the square root of $11+6\sqrt{2}$ is $\sqrt{9}+\sqrt{2}$, or $3+\sqrt{2}$.

Ex. 3. Required the square root of $11+2\sqrt{30}$.

Ans. $\sqrt{6}+\sqrt{5}$.

Ex. 4. Required the square root of $2+\sqrt{3}$.

Ans. $\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}$.

(279.) This method is applicable even when the binomial contains imaginary quantities, as in the following examples.

Ex. 5. Required the square root of $1+4\sqrt{-3}$.

Here $a=1$, and $\sqrt{b}=4\sqrt{-3}$; hence $b=-48$, and $a^2-b=49$; therefore, $c=7$. The required square root is $\sqrt{4}+\sqrt{-3}=2+\sqrt{-3}$.

Ex. 6. Required the square root of $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$.

Ans. $\frac{1}{2}+\frac{1}{2}\sqrt{-3}$.

Ex. 7. Required the square root of $2\sqrt{-1}$.

Here we put $a=o$; hence $c=2$, and the required root is

$$1+\sqrt{-1},$$

which may be easily verified.

Ex. 8. Required the square root of $6+2\sqrt{5}$ and $6-2\sqrt{5}$.

Ex. 9. Required the value of the expression

$$\sqrt{4+3\sqrt{-20}}+\sqrt{4-3\sqrt{-20}}.$$

Ans. 6.

SECTION XVIII.

INFINITE SERIES.

(280.) An *infinite series* is an infinite number of terms connected together by the signs + or —, and usually following some certain law.

As $\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}$, &c.,

or $\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}$, &c.

In the first of these examples, the terms are the reciprocals of the odd numbers, 1, 3, 5, 7, &c.; and in the other, they are the reciprocals of the even numbers, 2, 4, 6, 8, &c., with signs alternately + and —.

Infinite series may arise from the common operations of division, the extraction of roots, and other processes of calculation, as will be seen hereafter.

A converging series is one in which the terms continually decrease, as in the examples just given.

A diverging series is one in which the terms continually increase; as,

$$1-2+4-8+16-32+, \text{&c.}$$

An ascending series is one in which the powers of the unknown quantity continually increase; as,

$$1+ax+bx^2+cx^3+dx^4+ex^5+, \text{&c.}$$

A descending series is one in which the powers of the unknown quantity continually decrease; as,

$$1+ax^{-1}+bx^{-2}+cx^{-3}+dx^{-4}+ex^{-5}+, \text{&c.}$$

PROBLEM I.

(281.) *Any series being given, to find its several orders of differences.*

RULE.

1. *Take the first term from the second, the second from the third, the third from the fourth, &c.; and the remainders will form a new series, called the* FIRST ORDER OF DIFFERENCES.

2. *Take the first term of this last series from the second, the second from the third, the third from the fourth, &c.; and the remainders will form a third series, called the* SECOND ORDER OF DIFFERENCES.

3. *Proceed in like manner for the third, fourth, fifth, &c., orders of differences; and so on till they terminate, or are carried as far as may be thought necessary.*

Ex. 1. Required the several orders of differences of the series of squares, 1 4 9 16 25 36 49, &c.

3 5 7 9 11 13 first differences.
2 2 2 2 2 second differences.
0 0 0 0 third differences

Ex. 2. Required the several orders of differences of the series of cubes, 1 8 27 64 125 216, &c.

7 19 37 61 91 first differences.
12 18 24 30 second differences.
6 6 6 third differences.
0 0 fourth differences.

Ex. 3. Required the several orders of differences of the series of fourth powers,

1, 16, 81, 256, 625, 1296, &c.

Ex. 4. Required the several orders of differences of the series of fifth powers,

1, 32, 243, 1024, 3125, 7776, 16807, &c.

Ex. 5. Required the several orders of differences of the series of numbers,

1, 3, 6, 10, 15, 21, &c.

PROBLEM II.

(282.) *To find the* n*th term of the series*

$$a,\ b,\ c,\ d,\ e,\ \&c.,$$

when the differences of any order become at last equal to each other.

Take the proposed series, and subtract each term from the next succeeding one; we shall obtain the following series, which is called the *first order of differences:*

$$b-a,\ c-b,\ d-c,\ e-d,\ f-e,\ \&c.$$

Again, subtracting each term of this series from the next succeeding term, we find for the *second order of differences,*

$$c-2b+a,\ d-2c+b,\ e-2d+c,\ f-2e+d,\ \&c.$$

Subtracting, again, each term of the preceding series from its next succeeding term, we get the *third order of differences,*

$$d-3c+3b-a,\ e-3d+3c-b,\ f-3e+3d-c,\ \&c.$$

Subtracting again, we find for the *fourth order of differences,*

$$e-4d+6c-4b+a,\ f-4e+6d-4c+b,\ \&c.$$

And for the *fifth order of differences,*

$$f-5e+10d-10c+5b-a,\ \&c.$$

Let D', D'', D''', D^{iv}, &c., represent the first terms of the several orders of differences.

Then,

$D' = b-a;$ whence $b=a+D'$

$D'' = c-2b+a;$ whence $c=a+2D'+D''$

$D'''=d-3c+3b-a;$ whence $d=a+3D'+3D''+D'''$

$D^{iv} = e-4d+6c-4b+a;$ whence $e=a+4D'+6D''+4D'''+D^{iv}$,

&c., &c.

The coefficients of the value of c, the *third* term of the proposed series, are 1, 2, 1, which are the coefficients of the *second* power of a binomial; the coefficients of the value of d, the *fourth* term, are 1, 3, 3, 1, which are the coefficients of the *third* power of a binomial; and so on. Hence we infer that the coefficients of the

nth term of the series, are the coefficients of the $(n-1)$ power of a binomial. Therefore, the nth term of the series will be

$$a+\frac{(n-1)}{1}D'+\frac{(n-1)(n-2)}{1.2}D''+\frac{(n-1)(n-2)(n-3)}{1.2.3}D'''+,$$
&c.

Ex. 1. Required the twelfth term of the series,

2, 6, 12, 20, 30, &c.

The first order of differences is 4 6 8 10, &c.
The second order of differences is 2 2 2, &c.
The third order of differences is 0 0.

Therefore, $D'=4$, $D''=2$, and $D'''=0$. Also, $a=2$, and $n=12$.

Then $a+(n-1)D'+\dfrac{(n-1)(n-2)D''}{2}=2+11D'+55D''=$

$=2+44+110=156=$ the twelfth term.

Ex. 2. Required the twentieth term of the series,

1, 3, 6, 10, 15, 21, &c.

Here, $a=1$, $D'=2$, $D''=1$, and $n=20$
Therefore, the 20th term $=1+19D'+171D''=1+38+171$
$=210$, Ans.

Ex. 3. Required the thirteenth term of the series,

1, 2, 3, 6, 12, 24, 48, &c.

Ans. 75.

Ex. 4. Required the fifteenth term of the series,

1, 4, 9, 16, 25, 36, &c.

Ans. 225.

Ex. 5. Required the twentieth term of the series,

1, 8, 27, 64, 125, &c.

PROBLEM III.

(283.) *To find the sum of* n *terms of the series,*

a, b, c, d, e, &c.

Assume the series,

o, a, $a+b$, $a+b+c$, $a+b+c+d$, &c.

Subtracting each term from the next succeeding, we obtain

$$a,\ b,\ c,\ d,\ e,\ \&c.,$$

which is the series whose sum we propose to find. Hence the sum of n terms of the proposed series, is the $(n+1)$th term of the assumed series; and the nth order of differences in one series is the $(n+1)$th order in the other series. If, therefore, in the formula of the preceding Problem, we substitute

$$\begin{aligned} 0 &\text{ for } a, \\ n+1 &\text{ for } n, \\ a &\text{ for } D', \\ D' &\text{ for } D''. \\ &\&c., \end{aligned}$$

we shall have

$$na+\frac{n(n-1)}{2}D'+\frac{n(n-1)(n-2)}{2.3}D''+\frac{n(n-1)(n-2)(n-3)}{2.3.4}D'''$$
$$+, \&c.,$$

which is the sum of n terms of the proposed series.

Ex. 1. Required the sum of n terms of the series

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \&c.$$

Here $a=1,\ D'=1,\ D''=0.$

Therefore, $na+\frac{n(n-1)D'}{2}=n+\frac{n^2-n}{2}=\frac{n^2+n}{2}=\frac{n(n+1)}{2}=$ the sum of n terms, the same as found in Art. 239.

Ex. 2. Required the sum of n terms of the series

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ 5^2,\ \&c.$$

Here $a=1,\ D'=3,\ D''=2.$

Therefore the general formula reduces to

$$n+\frac{3n(n-1)}{2}+\frac{2n(n-1)(n-2)}{2.3.},$$
$$=\frac{2n^3+3n^2+n}{6},$$
$$=\frac{n(n+1)(2n+1)}{6}, \text{ the sum required.}$$

Ex. 3. Required the sum of n terms of the series

$$1^3,\ 2^3,\ 3^3,\ 4^3,\ 5^3,\ 6^3,\ \&c.$$

Here $a=1$, $D'=7$, $D''=12$, $D'''=6$.

Ans. $\dfrac{n^2(n+1)^2}{4}$.

Ex. 4. Required the sum of n terms of the series

$$1,\ 3,\ 6,\ 10,\ 15,\ \&c.$$

Ans. $\dfrac{n(n+1)(n+2)}{2.3}$.

Ex. 5. Required the sum of n terms of the series

$$1,\ 4,\ 10,\ 20,\ 35,\ \&c.$$

Ans. $\dfrac{n(n+1)(n+2)(n+3)}{2.3.4}$.

PROBLEM IV.

(284.) *Any series of equidistant terms*, a, b, c, d, e, *&c., being given, to find any intermediate term by interpolation.*

This is essentially the same as Problem II. For convenience, let us put x to represent the distance of the required term from the first term of the series a, in which case $x=n-1$, and we shall have

$$z=a+D'x+\frac{D''x(x-1)}{2}+\frac{D'''x(x-1)(x-2)}{2.3}+,\ \&c.$$

Ex. 1. Given the square root of 160, equal to 12.64911;
" " 162, " 12.72792;
" " 164, " 12.80625,

to find the square root of 161.

Here the first differences are +.07881, +.07833.
And the second difference is —.00048.

The interval between the given numbers is 2; the distance of the required term from the first term is 1; and, since this is always to be reckoned in parts of the equal intervals of the given numbers, we have $x=\frac{1}{2}$.

Also, $D'=+.07881$, $D''=-.00048$.

Hence $z=a+\frac{1}{2}D'-\frac{1}{8}D''$,
$=12.64911+.03941+.00006$,
$=12.68858$, which is the square root of 161.

Ex. 2. Given the cube root of 60, equal to 3.91487;
" " 62, " 3.95789;
" " 64, " 4.00000;
" " 66, " 4.04124,
to find the cube root of 61.

Ans. 3.93650.

Ex. 3. Given the fourth root of 625, equal to 5.000000;
" " 628, " 5.005988;
" " 631, " 5.011956;
" " 634, " 5.017903,
to find the fourth root of 627

Here $x = \frac{2}{3}$. Therefore, $z = a + \frac{2}{3}D' - \frac{1}{9}D''$.

Ans. 5.003994.

Ex. 4. Given the square root of 70, equal to 8.36660;
" " 74, " 8.60233;
" " 78, " 8.83176;
" " 82, " 9.05539,
to find the square root of 71.

Ans. 8.42615.

(285.) *Fractions expanded into infinite series.*

When the dividend is not divisible by the divisor, the quotient may be expressed by a fraction. Thus, if it is required to divide 1 by $1 - a$, we obtain the fraction $\frac{1}{1-a}$. We may, however, proceed with the division according to the usual method, thus:

$$\begin{array}{l|l}
1 & 1 - a \\ \cline{2-2}
\underline{1 - a} & 1 + a + a^2 + a^3 + a^4 +, \text{\&c.}, = \text{the quotient.} \\
\quad a & \\
\quad \underline{a - a^2} & \\
\qquad a^2 & \\
\qquad \underline{a^2 - a^3} & \\
\qquad\quad a^3 & \\
\qquad\quad \underline{a^3 - a^4} & \\
\qquad\qquad a^4. &
\end{array}$$

Hence $\frac{1}{1-a} = 1 + a + a^2 + a + a^4 + a^5 +$, &c., to infinity.

Suppose $a = \frac{1}{2}$, we shall then have

$\frac{1}{1-a} = \frac{1}{1-\frac{1}{2}} = 2$, which will be equal to the series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} +, \text{\&c.}$$

Suppose $a = \frac{1}{3}$, we shall then have

$\frac{1}{1-a} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$, which will be equal to the series

$$1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} +, \text{\&c.}$$

Ex. 2. Resolve $\frac{1}{1+a}$ into an infinite series.

Ans. $1 - a + a^2 - a^3 + a^4 - a^5 +$, &c.

Suppose $a = \frac{1}{2}$, we shall then have

$\frac{1}{1+\frac{1}{2}} = \frac{2}{3}$, which will be equal to the series

$$1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{16} - \tfrac{1}{32} +, \text{\&c.}$$

Ex. 3. Resolve the fraction $\frac{c}{a+b}$ into an infinite series.

Ans. $\frac{c}{a} - \frac{bc}{a^2} + \frac{b^2c}{a^3} - \frac{b^3c}{a^4} +$, &c.

Ex. 4. Resolve $\frac{a^2}{b+x}$ into an infinite series.

Ex. 5. Resolve $\frac{1+x}{1-x}$ into an infinite series.

We may proceed in the same manner when there are more than two terms in the divisor.

Ex. 6. Resolve $\frac{1}{1-a+a^2}$ into an infinite series

Ans. $1 + a - a^3 - a^4 + a^6 +$, &c.

Ex. 7. Resolve $\frac{a^2}{(a+x)^2}$ into an infinite series.

(286.) *Infinite series obtained by extracting the square root.*

In Art. 267, $\sqrt{a+b}$ has been expanded into an infinite series by the Binomial Theorem. It was also remarked that the same result might have been obtained by extracting the square root according to the usual rule. The operation will be as follows:

$$\begin{array}{l|l}
a+b & a^{\frac{1}{2}}+\frac{b}{2a^{\frac{1}{2}}}-\frac{b^2}{8a^{\frac{3}{2}}}+\frac{b^3}{16a^{\frac{5}{2}}}-\frac{5b^4}{128a^{\frac{7}{2}}}+,\&c.,=\text{square root of } a+b.\\
\underline{a} & \\
b & 2a^{\frac{1}{2}}+\frac{b}{2a^{\frac{1}{2}}},\ \text{first divisor.}\\
\underline{b+\frac{b^2}{4a}} & \\
-\frac{b^2}{4a} & 2a^{\frac{1}{2}}+\frac{b}{a^{\frac{1}{2}}}-\frac{b^2}{8a^{\frac{3}{2}}},\ \text{second divisor.}\\
\underline{-\frac{b^2}{4a}-\frac{b^3}{8a^2}+\frac{b^4}{64a^3}} & \\
+\frac{b^3}{8a^2}-\frac{b^4}{64a^3} & 2a^{\frac{1}{2}}+\frac{b}{a^{\frac{1}{2}}}-\frac{b^2}{4a^{\frac{3}{2}}}+\frac{b^3}{16a^{\frac{5}{2}}},\ \text{third divisor.}\\
\underline{\frac{b^3}{8a^2}+\frac{b^4}{16a^3}-\frac{b^5}{64a^4}+\frac{b^6}{256a^5}} & \\
-\frac{5b^4}{64a^3}+\frac{b^5}{64a^4}-\frac{b^6}{256a^5}. &
\end{array}$$

This result is the same as that obtained in Art. 267.

Ex. 2. Extract the square root of $1 + x$.

$$\text{Ans. } 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+,\ \&c.$$

Ex. 3. Extract the square root of $a^2 + b$.

Ex. 4. Extract the square root of $a^4 - b$.

METHOD OF UNKNOWN COEFFICIENTS.

(287.) The method of unknown coefficients is a method for the development of algebraic expressions, by assuming a series with unknown coefficients, and afterward finding the value of these coefficients. This method is founded on the following

THEOREM.

If an equation of the form

$$A+Bx+Cx^2+Dx^3+,\ \&c.,=A'+B'x+C'x^2+D'x^3+,\ \&c.,$$

must be verified by any value given to* x*, the terms involving the same powers in the two members are respectively equal.

For, since this equation must be verified for every value of x, it must be verified when $x = 0$. But, upon this supposition,

$$A = A'.$$

Suppressing these two equal terms, we have

$$Bx + Cx^2 + Dx^3 +, \&c., = B'x + C'x^2 + D'x^3 +, \&c.$$

Dividing every term by x, we have

$$B + Cx + Dx^2 +, \&c., = B' + C'x + D'x^2 +, \&c.$$

Since this equation must be verified for every value of x, it must be verified when $x = 0$. But, upon this supposition,

$$B = B'.$$

In the same manner. we can prove that

$$C = C',$$
$$D = D', \&c.$$

(288.) In order to give some idea of this method, we propose to develop the expression $\frac{1-x}{1+x}$ into a series arranged according to the powers of x. It is plain that this development is possible, for we may divide the numerator by the denominator, as explained in Art. 285.

Let us, then, assume

$$\frac{1-x}{1+x} = A + Bx + Cx^2 + Dx^3 + Ex^4 +, \&c.,$$

A, B, C, D, being coefficients independent of x, but dependent on the known terms of the fraction.

In order to obtain the values of these coefficients, let us multiply both members of the above equation by $1 + x$, and we shall have

$$1-x = A + (A+B)x + (B+C)x^2 + (C+D)x^3 + (D+E)x^4 +, \&c.$$

But, according to the preceding Theorem, the terms involving the same powers of x in the two members of the equation must be equal to each other.

Therefore,
$$\begin{array}{rll} A = & 1, & \\ A + B = & -1; & \text{hence } B = -2. \\ B + C = & 0; & \text{"} \quad C = +2. \\ C + D = & 0; & \text{"} \quad D = -2. \\ D + E = & 0; & \text{"} \quad E = +2, \\ \&c., & & \&c. \end{array}$$

Substituting these values of the coefficients in the assumed series, we obtain

$$\frac{1-x}{1+x} = 1 - 2x + 2x^2 - 2x^3 + 2x^4 -, \&c.$$

(289.) The method thus exemplified is expressed in the following

RULE.

Assume a series with unknown coefficients for that which is required to be found; then, having multiplied it by the denominator of the given fraction, or raised it to its proper power, find the value of each of these coefficients by equating the corresponding terms of the two expressions, or putting such of them as have no corresponding terms, equal to zero.

Ex. 2. Expand the fraction $\frac{1}{1-2x+x^2}$ into an infinite series.

Assume $\frac{1}{1-2x+x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 +, \&c.$

Multiplying by $1 - 2x + x^2$, we have

$$1 = A + (B - 2A)x + (C - 2B + A)x^2 + (D - 2C + B)x^3 + (E - 2D + C)x^4 +, \&c.$$

Hence we must have

$$\begin{array}{rl} A = 1 & \\ B - 2A = 0 & \because B = 2A \qquad = \quad , \\ C - 2B + A = 0 & \because C = 2B - A = 3, \\ D - 2C + B = 0 & \because D = 2C - B = 4, \\ E - 2D + C = 0 & \because E = 2D - C = 5, \\ \&c., & \qquad \&c \end{array}$$

Therefore, $\frac{1}{1-2x+x^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 +, \&c.$

Ex. 3. Expand the fraction $\frac{1+2x}{1-x-x^2}$ into an infinite series.

$$\text{Ans. } 1+3x+4x^2+7x^3+11x^4+18x^5+29x^6+, \text{ \&c.},$$

where the coefficient of each term is equal to the sum of the coefficients of the two preceding terms.

Ex. 4. Expand $\frac{1-x}{1-2x-3x^2}$ into an infinite series.

$$\text{Ans. } 1+x+5x^2+13x^3+41x^4+121x^5+, \text{ \&c.}$$

Ex. 5. Expand $\frac{1+2x}{1-3x}$ into an infinite series.

$$\text{Ans. } 1+5x+15x^2+45x^3+135x^4+, \text{ \&c.}$$

Ex. 6. Expand $\sqrt{1-x}$ into an infinite series.

$$\text{Ans. } 1-\frac{x}{2}-\frac{x^2}{2.4}-\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}-\frac{3.5.7x^5}{2.4.6.8.10}-, \text{ \&c.}$$

(290.) The method of unknown coefficients requires that we should know beforehand the *form* of the development, with respect to the powers of x. Generally, we suppose the development to proceed according to the ascending powers of x, commencing with x^0; but sometimes this form is *inapplicable*, in which case the result of the operation is sure to indicate it.

Let it be required, for example, to develop the expression $\frac{1}{3x-x^2}$ into a series.

Assume $$\frac{1}{3x-x^2}=\text{A}+\text{B}x+\text{C}x^2+\text{D}x^3+, \text{ \&c.}$$

Clearing of fractions, we have

$$1=3\text{A}x+(3\text{B}-\text{A})x^2+(3\text{C}-\text{B})x^3+, \text{ \&c.};$$

whence, according to Art. 287, we conclude

$$1=0,$$
$$3\text{A}=0, \text{ \&c.}$$

Now the first equation, $1=0$, is *absurd*, and shows that the assumed form is not applicable in the present case. But if we put the fraction under the form $\frac{1}{x}\times\frac{1}{3-x}$, and suppose that

$$\frac{1}{x}\times\frac{1}{3-x}=\frac{1}{x}(\text{A}+\text{B}x+\text{C}x^2+\text{D}x^3+, \text{ \&c.}),$$

it will become, after the reductions are made,

$$1 = 3A + (3B - A)x + (3C - B)x^2 + (3D - C)x^3 +, \&c.,$$

which gives the equations

$$\begin{aligned} 3A &= 1; \text{ whence } A = \tfrac{1}{3}. \\ 3B - A &= 0; \quad \text{"} \quad B = \tfrac{1}{9}. \\ 3C - B &= 0; \quad \text{"} \quad C = \tfrac{1}{27}. \\ 3D - C &= 0; \quad \text{"} \quad D = \tfrac{1}{81}. \end{aligned}$$

Therefore, $\dfrac{1}{3x - x^2} = \dfrac{1}{x}\left(\dfrac{1}{3} + \dfrac{x}{9} + \dfrac{x^2}{27} + \dfrac{x^3}{81} +, \&c.\right)$

$$= \frac{x^{-1}}{3} + \frac{x^0}{9} + \frac{x}{27} + \frac{x^2}{81} +, \&c.,$$

that is, the development contains a term affected with a *negative exponent.*

We ought, then, to have assumed at the outset

$$\frac{1}{3x - x^2} = Ax^{-1} + B + Cx + Dx^2 + Ex^3 +, \&c.$$

The particular series which is to be adopted in each case may be determined by putting $x = 0$, and observing the nature of the result. If, in this case, the expression to be developed becomes equal to a finite quantity, the first term of the series will not contain x. If the expression reduces to zero, the first term will contain x; and if the expression reduces to the form $\frac{A}{0}$, then the first term of the development must contain x with a negative exponent.

SECTION XIX.

GENERAL THEORY OF EQUATIONS.

(291.) It is proposed in this Section to exhibit the most important propositions relating to the theory of equations, together with the Theorem of Sturm, by which we are enabled to determine the number of real roots of an equation.

A *function* of a quantity is any expression involving that quantity. Thus,

ax^2+b is a function of x.

ay^3+cy+d is a function of y.

ax^2-by^2 is a function of x and y.

In a series of terms, two successive signs constitute a *permanence* when the signs are alike; and a *variation* when they are unlike. Thus, in the polynomial

$$a+b-c+d,$$

the signs of the first two terms constitute a permanence; the signs of the second and third constitute a variation; and those of the third and fourth also a variation.

(292.) A *cubic equation* is one in which the highest power of the unknown quantity is of the third degree, as, for example,

$$x^3-6x^2+8x-15=0.$$

All equations of the third degree may be reduced to the form

$$x^3+ax^2+bx+c=0.$$

A *biquadratic equation* is one in which the highest power of the unknown quantity is of the fourth degree; as, for example,

$$x^4-6x^3+7x^2+5x-4=0.$$

Every equation of the fourth degree may be reduced to the form

$$x^4+ax^3+bx^2+cx+d=0.$$

The general form of an equation of the fifth degree is

$$x^5+ax^4+bx^3+cx^2+dx+e=0;$$

and the general form of an equation of the *m*th degree is

$$x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}+\ldots\ldots+Tx+V=0 \quad (m).$$

This equation will be frequently referred to hereafter by the name of *the general equation of the* m*th degree*, or simply by the letter (m).

It is obvious, that if we could solve this equation, we should have the solution of every equation which could be proposed. Unfortunately, no general solution has ever been discovered; yet many important properties are known which enable us to solve any numerical equation which can ever occur.

PROPOSITION I.

(293.) *If* a *is a root of the general equation of the* m*th degree, the equation will be exactly divisible by* x — a.

For if a is one value of x, the equation must be verified when we substitute a in the place of x. Hence we must have

$$a^m+Aa^{m-1}+Ba^{m-2}+Ca^{m-3}+\ldots\ldots+Ta+V=0 \quad (1).$$

Subtracting equation (1) from equation (m), we obtain

$$(x^m-a^m)+A(x^{m-1}-a^{m-1})+B(x^{m-2}-a^{m-2})+\ldots+T(x-a)=0 \quad (2).$$

But, by Art. 76, each of the expressions (x^m-a^m), $(x^{m-1}-a^{m-1})$, &c., is divisible by $x-a$; and therefore equation (2) is also divisible by $x-a$. Now equation (m) is essentially the same as equation (2), for if we take the value of V, as found from equation (1), and substitute it for V in equation (m), it will give us equation (2); therefore, equation (m) is divisible by $x-a$.

Conversely, if equation (m) is divisible by $x-a$, then a is a root of the equation.

It will be noticed that this property is but a *generalization* of what has been proved of equations of the second degree in Art. 192.

Ex. 1. Prove that 1 is a root of the equation

$$x^3-6x^2+11x-6=0.$$

This equation is divisible by $x-1$, and gives $x^2-5x+6=0$.

Ex. 2. Prove that 2 is a root of the equation

$$x^3-x-6=0.$$

This equation is divisible by $x-2$, and gives $x^2+2x+3=0$.

Ex. 3. Prove that 2 is a root of the equation

$$x^3-11x^2+36x-36=0.$$

Ex. 4. Prove that 4 is a root of the equation

$$x^3+x^2-34x+56=0.$$

Ex. 5. Prove that -1 is a root of the equation

$$x^4-38x^3+210x^2+538x+289=0.$$

Ex. 6. Prove that -5 is a root of the equation

$$x^5+6x^4-10x^3-112x^2-207x-110=0.$$

Ex. 7. Prove that 3 is a root of the equation

$$x^7+x^6-14x^5-14x^4+49x^3+49x^2-36x-36=0.$$

PROPOSITION II.

(294.) *Every equation containing but one unknown quantity, has as many roots as there are units in the highest power of the unknown quantity, and no more.*

For, suppose a to be a root of the general equation of the mth degree. By the last Proposition, this equation is divisible by $x-a$; and if we actually perform the division, the equation will be reduced to one of the next inferior degree, and may be represented by the form

$$x^{m-1}+A'x^{m-2}+B'x^{m-3}+\ldots\ldots+T'x+V'=0$$

This equation must also have a root, which we will represent by b; and dividing by $x-b$, the equation will be reduced to one of the next inferior degree, and so on.

We may continue this operation $(m-1)$ times, when we shall arrive at a simple equation which has only one root. Hence, the proposed equation will have m roots,

$$a, b, c, d, \ldots\ldots l;$$

and its successive divisors, or the factors of which it is composed, will be

$$x-a,\ x-b,\ x-c,\ x-d, \ldots\ldots x-l,$$

being equal in number to the units contained in m, the index of the highest term of the equation.

We have seen that when one root of an equation is known, the depressed equation containing the remaining roots is readily found by division; and if we can depress any equation to a quadratic, its roots can be determined by methods already explained.

Ex. 1. One root of the equation

$$x^3+3x^2-16x+12=0$$

is 1. Find the remaining roots.

Ex. 2. Two roots of the equation

$$x^4-25x^2+60x-36=0$$

are 1 and 3. Find the remaining roots.

Ex. 3. Two roots of the equation

$$x^4-12x^3+48x^2-68x+15=0$$

are 3 and 5. Find the remaining roots.

Ex. 4. Two roots of the equation

$$4x^4-14x^3-5x^2+31x+6=0$$

are 2 and 3. Find the remaining roots.

Ex. 5. Two roots of the equation

$$x^4-6x^3+24x-16=0$$

are 2 and -2. Find the remaining roots.

(295.) It should be observed that this Proposition only proves that an equation of the mth degree may be continually depressed by division, and finally exhausted after m operations. The divisors are not necessarily *unequal. Any number, and indeed all of them, may be equal.* When we say that an equation of the mth degree has m roots, we mean that the polynomial can be decomposed into m binomial factors, equal or unequal, each containing one root. Thus, the equation

$$x^3-6x^2+12x-8=0$$

can be resolved into the factors

$$(x-2)\,(x-2)\,(x-2)=0\,;\text{ or }(x-2)^3=0\,;$$

whence it appears that the three roots of this equation are

$$2,\ 2,\ 2.$$

But, in general, the several roots of an equation differ from each other numerically.

The equation

$$x^3=8$$

has apparently but one root, viz., 2; but by the method of the preceding article we can discover two other roots. Dividing x^3-8 by $x-2$, we obtain

$$x^2+2x+4=0.$$

Solving this equation, we find

$$x=-1\pm\sqrt{-3}.$$

Thus, the three roots of the equation $x^3=8$ are

$$2\,;\ -1+\sqrt{-3}\,;\ -1-\sqrt{-3}.$$

These last two values may be verified by multiplication as follows:

$$\begin{array}{lcl}
\begin{array}{r}
-1+\sqrt{-3} \\
-1+\sqrt{-3} \\ \hline
1-\sqrt{-3} \\
-\sqrt{-3}-3 \\ \hline
-2-2\sqrt{-3} \\
-1+\sqrt{-3} \\ \hline
2+2\sqrt{-3} \\
-2\sqrt{-3}+6 \\ \hline
8
\end{array}
& &
\begin{array}{r}
-1-\sqrt{-3} \\
-1-\sqrt{-3} \\ \hline
1+\sqrt{-3} \\
+\sqrt{-3}-3 \\ \hline
-2+2\sqrt{-3} \\
-1-\sqrt{-3} \\ \hline
2-2\sqrt{-3} \\
+2\sqrt{-3}+6 \\ \hline
8
\end{array}
\end{array}$$

$-2-2\sqrt{-3}=$ the square. $\quad -2+2\sqrt{-3}=$ the square.

$8=$ the cube. $\quad 8=$ the cube.

If the last term of an equation vanishes, as in the example

$$x^4 + 2x^3 + 3x^2 + 6x = 0,$$

the equation is divisible by $x - 0$; and consequently 0 is one of its roots.

If the last two terms vanish, then two of its roots are equal to 0.

PROPOSITION III.

Law of the coefficients of every equation.

(296.) In order to discover the law of the coefficients, let us form the equation whose roots are

$$a, b, c, d, \ldots\ldots l.$$

This equation will contain the factors $(x - a)$, $(x - b)$, $(x - c)$, &c.; that is, we shall have

$$(x - a)(x - b)(x - c)(x - d) \ldots\ldots (x - l) = 0.$$

If we perform the multiplication as in Article 259, we shall have

$$\begin{array}{r|r|r|l}
x^m - a & x^{m-1} + ab & x^{m-2} - abc & x^{m-3} + \ldots\ldots - (abc\ldots\ldots l) = 0. \\
- b & + ac & - abd & \\
- c & + ad & - acd & \\
- d & + bc & - bcd & \\
\&c & + bd & \&c. & \\
 & + cd & & \\
 & \&c. & &
\end{array}$$

Hence we perceive,

1. *The coefficient of the second term of any equation is equal to the sum of all the roots with their signs changed.*

2. *The coefficient of the third term is equal to the sum of the products of all the roots taken two and two.*

3. *The coefficient of the fourth term is equal to the sum of the products of all the roots taken three and three, with their signs changed.*

4. *The last term is the product of all the roots with their signs changed.*

It will be perceived that these properties include those of quadratic equations mentioned on page 195.

If the roots are all negative, the signs of all the terms of the equation will be positive, because the factors of which the equation is composed are all positive.

If the roots are all positive, the signs of the terms will be alternately + and —.

Ex. 1. Form the equation whose roots are 1, 2, and 3.

For this purpose, we must multiply together the factors $x-1$, $x-2$, $x-3$, and we obtain

$$x^3-6x^2+11x-6=0.$$

This example conforms to the rules above given for the coefficients. Thus, the coefficient of the second term is equal to the sum of all the roots $(1+2+3)$ with their signs changed.

The coefficient of the third term is the sum of the products of the roots taken two and two; thus,

$$1\times2+1\times3+2\times3.$$

The last term is the product of all the roots $(1\times2\times3)$ with their signs changed.

Ex. 2. Form the equation whose roots are 2, 3, 5, and -6.

Ans. $x^4-4x^3-29x^2+156x-180=0$.

Show how these coefficients conform to the laws above given.

Ex. 3. Form the equation whose roots are 1, 3, 5, -2, -4, -6.

Ans. $x^6+3x^5-41x^4-87x^3+400x^2+444x-720=0$.

(297.) *Every rational root of an equation is a divisor of the last term;* for, since this term is the product of all the roots, it must be *divisible* by each of them. If, then, we wish to find a root by trial, we know at once what numbers we must employ.

For example, take the equation

$$x^3-x-6=0.$$

If this equation has a rational root, it must be a divisor of the last term 6; hence we must try the numbers 1, 2, 3, 6.

$$\begin{array}{llll} \text{If} & x=1, \text{ we have} & 1-1-6= & -6, \\ & x=2, \text{ “ “} & 8-2-6= & 0, \\ & x=3, \text{ “ “} & 27-3-6= & 18, \\ & x=6, \text{ “ “} & 216-6-6= & 204, \end{array}$$

Hence we see that 2 is one of the roots of the given equation, and by the method of Art. 294, we shall find the remaining roots to be

$$-1 \pm \sqrt{-2}.$$

PROPOSITION IV.

(298.) *No equation whose coefficients are all integers, and that of the highest power of the unknown quantity unity, can have a fractional root.*

For, take the equation

$$x^3 + Ax^2 + Bx + C = 0,$$

and suppose, if possible, that the irreducible fraction $\frac{a}{b}$ is one value of x. If we substitute this value for x in the given equation, we shall have

$$\frac{a^3}{b^3} + A\frac{a^2}{b^2} + B\frac{a}{b} + C = 0.$$

Multiplying each term by b^2, and transposing, we obtain

$$\frac{a^3}{b} = -(Aa^2 + Bab + Cb^2).$$

Now by supposition, A, B, C, a and b are whole numbers. Hence the entire right-hand member of the equation is a whole number.

But by hypothesis, $\frac{a}{b}$ is an irreducible fraction; that is, a and b contain no common factor. Consequently, a^3 and b will contain no common factor, that is, $\frac{a^3}{b}$ is a fraction in its lowest terms. Hence, if $\frac{a}{b}$ were a root of the proposed equation, we should have a fraction in its lowest terms equal to a whole number, which is *absurd.*

The same mode of demonstration is applicable to the general equation of the mth degree.

PROPOSITION V.

(299.) *If the signs of the alternate terms in an equation be changed, the signs of all the roots will be changed.*

If we take the general equation of the *m*th degree, and change the signs of the alternate terms, we shall have

$$x^m - Ax^{m-1} + Bx^{m-2} - Cx^{m-3} + \ldots\ldots = 0 \quad (1);$$

or, changing the sign of every term of the last equation,

$$-x^m + Ax^{m-1} - Bx^{m-2} + Cx^{m-3} - \ldots\ldots = 0 \quad (2).$$

Now, substituting $+a$ for x in equation (m) will give the same result as substituting $-a$ in equation (1), if m be an *even* number; or, substituting $-a$ in equation (2), if m be an *odd* number. If, then, a is a root of equation (m), $-a$ will be a root of equation (1), and of course a root of equation (2), which is identical with it.

Hence we see that the positive roots may be changed into negative roots, and the reverse, by simply changing the signs of the alternate terms; so that the finding the real roots of any equation is reduced to finding positive roots only.

Ex. 1. The roots of the equation

$$x^3 - 2x^2 - 5x + 6 = 0$$

are 1, 3, and -2. What are the roots of the equation

$$x^3 + 2x^2 - 5x - 6 = 0 ?$$

Ex. 2. The roots of the equation

$$x^3 - 6x^2 + 11x - 6 = 0$$

are 1, 2, and 3. What are the roots of the equation

$$x^3 + 6x^2 + 11x + 6 = 0 ?$$

PROPOSITION VI.

(300.) *If an equation whose coefficients are all real, contains imaginary roots, the number of these roots must be even.*

If an equation whose coefficients are all real, has a root of the form

$$a + b\sqrt{-1},$$

then will

$$a - b\sqrt{-1}$$

be also a root of the equation.

For, from Art. 296, the *product* of all the roots must be equal to a real quantity. Now the only factor which will render $a+b\sqrt{-1}$ real, is $a-b\sqrt{-1}$, or $ac-bc\sqrt{-1}$. But the *sum* of all the roots must also be equal to a real quantity; therefore the imaginary parts of the two roots must be equal, and have opposite signs.

Ex. 1. Find the roots of the equation

$$x^3-2x+4=0.$$

Ans. $-2,\ 1\pm\sqrt{-1}$.

Ex. 2. Find the roots of the equation

$$x^3-x^2-7x+15=0.$$

Ans. $-3,\ 2\pm\sqrt{-1}$.

Ex. 3. Find the roots of the equation

$$5x^3+2x-44=0.$$

Ans. $2,\ -1\pm\sqrt{-3.4}$

PROPOSITION VII.

(301.) *Every equation must have as many variations of sign as it has positive roots; and as many permanences of sign as there are negative roots.*

To prove this Proposition, it is only necessary to show that the multiplication of an equation by a new factor, $x-a$, corresponding to a *positive* root, will introduce at least *one variation*; and that the multiplication by a factor $x+a$ will introduce at least *one permanence.*

For an example, take the equation

$$x^3+3x^2-10x-24=0,$$

in which the signs are + + — —, giving *one variation.*

Multiply this equation by $x-2=0$, as follows:

$$\begin{array}{lllll}
x^3 & +3x^2 & -10x & -24 & \\
x & -2 & & & \\
\hline
x^4 & +3x^3 & -10x^2 & -24x & \\
 & -2x^3 & -6x^2 & +20x & +48 \\
\hline
x^4 & +x^3 & -16x^2 & -4x & +48=0.
\end{array}$$

In this last product the signs are $+ + - - +$, giving *two variations;* that is, the introduction of a positive root has introduced one new variation in the signs of the terms.

To generalize this reasoning, we perceive that the signs in the upper line of the partial products are the *same* as in the given equation; but those in the lower line are all *contrary* to those of the given equation, and advanced one term toward the right.

Now, as long as each coefficient of the upper line is greater than the corresponding one in the lower, it will determine the sign of every term of the product. Hence, in this case, there will be in the product, with the exception of the last term, the same *changes* of sign as in the given equation. But the last term introduces a new variation, since its sign is contrary to that which immediately precedes it.

When a term in the lower line is larger than the corresponding one in the upper line, and has the contrary sign, there is a change from a permanence to a variation; for the lower sign is always contrary to the preceding upper sign. Hence, whenever we are obliged to descend from the upper to the lower line in order to determine the sign of the product, there is a variation which is not found in the proposed equation; and as all the remaining signs of the lower line are contrary to the preceding ones of the upper line, there must be the same *changes* of sign in this line as in the proposed equation. If we are obliged to reascend to the upper line, we may suppose the result to be either a variation or a permanence. But even if it were a permanence, since the last sign of the product is in the lower line, it is necessary to go once more from the upper line to the lower, than from the lower to the upper. Hence each factor, corresponding to a positive root, must introduce at least *one new variation;* so that there must be as many variations as there are positive roots.

In the same manner, we may prove that the multiplication by a factor $x + a$, corresponding to a *negative* root, must introduce at least one new *permanence;* so that there must be as many permanences as there are negative roots.

Ex. 1. The roots of the equation

$$x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12 = 0$$

are 1, 2, 3, -1, and -2. There are also three variations of sign, and two permanences, as there should be, according to the Proposition.

Ex. 2. The equation

$$x^4 - 3x^3 - 15x^2 + 49x - 12 = 0$$

has four real roots. How many of these are negative ?

Ex. 3. The equation

$$x^6 + 3x^5 - 41x^4 - 87x^3 + 400x^2 + 444x - 720 = 0$$

has six real roots. How many of these are positive ?

If all the roots of an equation are *real*, the number of positive roots must be the same as the number of variations, and the number of negative roots must be the same as the number of permanences.

PROPOSITION VIII.

(302.) *Any equation may be transformed into another, whose roots shall be greater or less than those of the former by any given quantity.*

Let it be required to transform the general equation of the mth degree into another whose roots shall be greater by r than those of the given equation.

Take $y = x + r$, or $x = y - r$,

and substitute $y - r$ for x in the proposed equation ; we shall have

$$\begin{array}{lr|lr|lr|l}
y^m & -mr & y^{m-1} & +\frac{m(m-1)r^2}{2} & y^{m-2} & -\frac{m(m-1)(m-2)r^3}{2.3} & y^{m-3}, \&c., =0, \\
 & +A & & -(m-1)Ar & & +\frac{(m-1)(m-2)Ar^2}{2} & \\
 & & & +B & & -(m-2)Br & \\
 & & & & & +C &
\end{array}$$

which equation evidently fulfils the required conditions, since y is greater than x by r.

If we take $y = x - r$, or $x = y + r$, we shall obtain in the same way an equation whose roots are *less* than those of the given equation by r.

Ex. 1. Find the equation whose roots are greater by 1 than those of the equation

$$x^3 + 3x^2 - 4x + 1 = 0.$$

Ans. $y^3 - 7y + 7 = 0$.

Ex. 2. Find the equation whose roots are less by 1 than those of the equation

$$x^3 - 2x^2 + 3x - 4 = 0.$$

Ans. $y^3 + y^2 + 2y - 2 = 0$.

Ex. 3. Find the equation whose roots are greater by 3 than those of the equation

$$x^4 + 9x^3 + 12x^2 - 14x = 0.$$

Ans. $y^4 - 3y^3 - 15y^2 + 49y - 12 = 0$.

Ex. 4. Find the equation whose roots are less by 2 than those of the equation

$$5x^4 - 12x^3 + 3x^2 + 4x - 5 = 0.$$

Ans. $5y^4 + 28y^3 + 51y^2 + 32y - 1 = 0$.

Ex. 5. Find the equation whose roots are less by 2 than those of the equation

$$x^5 + 2x^3 - 6x^2 - 10x + 8 = 0.$$

Ans. $y^5 + 10y^4 + 42y^3 + 86y^2 + 70y + 4 = 0$.

PROPOSITION IX.

(303.) *Any complete equation may be transformed into another whose second term is wanting.*

Since r in the preceding Proposition is indeterminate, we may put $-mr + A$ equal to zero, which will cause the second term of the general development to disappear. Hence $r = \frac{A}{m}$, and $x = y - \frac{A}{m}$.

Hence, to remove the second term of an equation, *substitute for the unknown quantity a new unknown quantity, together with such a part of the coefficient of the second term, taken with a contrary sign, as is denoted by the degree of the equation.*

Ex. 1. Transform the equation

$$x^3 - 6x^2 + 8x - 2 = 0$$

into another whose second term is wanting.

Here we take a new unknown quantity, and annex to it a third part of the coefficient of the second term of the equation with its sign changed; that is, we put $x=y+2$. Making this substitution, we obtain

$$y^3-4y-2=0, \text{ Ans.}$$

Ex. 2. Transform the equation

$$x^4-16x^3-6x+15=0$$

into another whose second term is wanting.

Here we put $x=y+4$.

$$\text{Ans. } y^4-96y^2-518y-777=0.$$

Ex. 3. Transform the equation

$$x^5+15x^4+12x^3-20x^2+14x-25=0$$

into another whose second term is wanting.

$$\text{Ans. } y^5-78y^3+412y^2-757y+401=0.$$

PROPOSITION X.

(304.) *If two numbers, when substituted for the unknown quantity in an equation, give results with contrary signs, there must be at least one root comprised between those numbers.*

Take, for example, the equation

$$x^3-2x^2+3x-44=0.$$

If we substitute 3 for x in this equation, we obtain -26; and if we substitute 5 for x, we obtain $+46$. There must, therefore, be a real root between 3 and 5; for, when we suppose $x=3$,

$$x^3+3x<2x^2+44.$$

But when we suppose $x=5$,

$$x^3+3x>2x^2+44.$$

Now both the quantities

$$x^3+3x \text{ and } 2x^2+44$$

increase while x increases. And since the first of these quantities, which was originally less than the second, has become the greater, it must increase more rapidly than the second. There must, therefore, be a point at which the two magnitudes are equal, and that value of x which renders these two magnitudes equal must be a root of the proposed equation.

In the same manner, it may be proved that *if any quantity* p, *and every quantity greater than* p, *substituted in an equation, renders the result positive, then* p *is greater than the greatest root.*

Hence also, if the signs of the alternate terms are changed, and *if* q, *and every quantity greater than* q, *renders the result positive, then* — q *is less than the least root.*

If the two numbers, which give results with contrary signs, differ from each other only by unity, it is plain that we have found the *integral part* of the root.

Ex. 1. Find the integral part of one of the roots of the equation

$$2x^4 - 11x^2 + 8x - 16 = 0.$$

When $x = 2$, the equation reduces to -12; and when $x = 3$, it reduces to $+71$. Hence there must be a root between 2 and 3; that is, 2 is the first figure of one of the roots.

Ex. 2. Find the first figure of one of the roots of the equation

$$x^3 + x^2 + x - 100 = 0.$$

Ans. 4.

Ex. 3. Find the first figure of one of the roots of the equation

$$x^3 - 4x^2 - 6x + 8 = 0.$$

PROPOSITION XI.

(305.) *Of Derived Polynomials.*

If we substitute $y + r$ for x in the general equation of the mth degree, we shall find the coefficients of r follow a remarkable law.

The equation, before it is developed, will be

$$(y+r)^m + A(y+r)^{m-1} + B(y+r)^{m-2} + \ldots\ldots + T(y+r) + V = 0.$$

If we actually involve the several terms $(y + r)^m$, $(y + r)^{m-1}$, &c., as was done in Art. 302, we shall obtain certain terms independent of r, others which contain the first power of r, others the second power of r, and so on; and the development will be of the following form:

$$X + X_{,}r + \frac{X_{,,}}{2}r^2 + \frac{X_{,,,}}{2.3}r^3 + \frac{X_{,,,,}}{2.3.4}r^4 + \ldots\ldots + r^m,$$

where the values of X, $X_{,}$, $X_{,,}$, &c., are

$$\begin{aligned} X &= y^m + Ay^{m-1} + By^{m-2} + Cy^{m-3} + \quad \ldots\ldots + Ty + V. \\ X_{,} &= my^{m-1} + (m-1)Ay^{m-2} + (m-2)By^{m-3} + \ldots\ldots + 2Sy + T. \\ X_{,,} &= m(m-1)y^{m-2} + (m-1)\,(m-2)Ay^{m-3} + \ldots\ldots \end{aligned}$$

Each of these polynomials may be derived from that immediately preceding it, by multiplying each term by the index of y *in that term, and diminishing the index by unity.*

The expressions $X_{,}$, $X_{,,}$, &c., are called derived polynomials of X. $X_{,}$ is called the *first derived polynomial*, $X_{,,}$ the *second derived polynomial*, $X_{,,,}$ the *third*, and so on.

Ex. 1. Find the equation whose roots are less by r than those of the equation

$$x^2 - 5x + 6 = 0.$$

Here we shall have

$$\begin{aligned} X &= y^2 - 5y + 6, \\ X_{,} &= 2y - 5, \\ X_{,,} &= 2, \\ X_{,,,} &= 0. \end{aligned}$$

But we have seen that when $y + r$ is substituted for x, the equation reduces to the form

$$X + X_{,}r + \frac{X_{,,}}{2}r^2 + \frac{X_{,,,}}{2.3}r^3 +, \text{\&c.}$$

Substituting the values of X, $X_{,}$, $X_{,,}$, &c., above found, we obtain

$$(y^2 - 5y + 6) + (2y - 5)r + r^2,$$

which is the development of

$$(y + r)^2 - 5(y + r) + 6.$$

Ex. 2. Find the equation whose roots are less by r than those of the equation

$$x^3 - 7x^2 + 8x - 3 = 0.$$

Here we shall have

$$\begin{aligned} X &= y^3 - 7y^2 + 8y - 3, \\ X_{,} &= 3y^2 - 14y + 8 \\ X_{,,} &= 6y - 14, \\ X_{,,,} &= 6, \\ X_{,,,,} &= 0; \end{aligned}$$

and, substituting these values in the same formula as above, we obtain

$$(y^3 - 7y^2 + 8y - 3) + (3y^2 - 14y + 8)r + \frac{1}{2}(6y - 14)r^2 + \frac{1}{2.3}6r^3,$$

which is the development of

$$(y + r)^3 - 7(y + r)^2 + 8(y + r) - 3.$$

Ex. 3. Find the successive derived polynomials of the equation

$$x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$$

Ex. 4. Find the successive derived polynomials of the equation

$$x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2 = 0.$$

PROPOSITION XII.

(306.) *Of equal roots.*

We have seen, in Art. 295, that an equation may have two or more *equal roots*. Thus, the equation

$$x^3 - 6x^2 + 12x - 8 = 0,$$

or

$$(x - 2)^3 = 0,$$

has the three equal roots 2, 2, 2. Such an equation and its first derived polynomial always contain a *common divisor;* for the first derived polynomial of the above equation is

$$3x^2 - 12x + 12,$$

or

$$3(x - 2)^2,$$

where it is evident that $(x - 2)^2$ is a common divisor of both equations. And, generally, if the given equation and its first derived polynomial have a common divisor of the form $(x - a)^m$, the equation will have $(m + 1)$ roots, each equal to a.

To determine, therefore, whether an equation has equal roots, *find the greatest common divisor between the equation and its first de-*

rived polynomial. If there is no common divisor, the equation has no equal roots. If there is a common divisor, this must be a function of x. *We then solve the equation formed by putting this equal to zero, and thus obtain all the equal roots.*

Ex. 1. Find the equal roots of the equation

$$x^3 - 8x^2 + 21x - 18 = 0.$$

The derived polynomial of this equation is

$$3x^2 - 16x + 21.$$

The greatest common divisor between this and the given equation is

$$x - 3.$$

Hence the equation has two roots, each equal to 3.

Ex. 2. Find the equal roots of the equation

$$x^3 - 13x^2 + 55x - 75 = 0.$$

Ex. 3. Find the equal roots of the equation

$$x^3 - 7x^2 + 16x - 12 = 0.$$

Ex. 4. Find the equal roots of the equation

$$x^4 - 6x^2 - 8x - 3 = 0.$$

PROPOSITION XIII.

(307.) *To find the number of real and imaginary roots of any equation.*

In 1829, M. Sturm discovered a theorem which determines the precise number of real roots, and of course the number of imaginary ones; since the real and imaginary roots are together equal in number to the degree of the equation. We propose now to develop this theorem.

Let X represent the first member of the general equation of the *m*th degree, which we suppose to have no equal roots, and let $X_{,}$ be its first derived polynomial, found by the method of Art. 305.

Divide X by $X_{,}$ until the remainder is of a lower degree than the divisor, and call this remainder $-X_{,,}$; that is, let $X_{,,}$ designate the remainder with a contrary sign. Divide $X_{,}$ by $X_{,,}$ in the same manner, and so on, designating the successive remainders with contrary signs by $X_{\prime\prime\prime}$, $X_{\prime\prime\prime\prime}$, &c., until the division terminates by leaving a numerical remainder independent of x; which must always be the case, according to the preceding Proposition, since the equation having no equal roots, there can be no factor, which is a function of x, common to the equation and its first derived polynomial. Let this remainder, having its signs changed, be called X_m.

The operation thus described will stand as follows:

$$\begin{array}{r|l} X & X_{\prime} \\ \underline{X_{\prime}Q_{\prime}} & \overline{Q_{\prime}} \\ \multicolumn{2}{l}{X - X_{\prime}Q_{\prime} = -X_{\prime\prime};} \end{array} \quad \begin{array}{r|l} X_{\prime} & X_{\prime\prime} \\ \underline{X_{\prime\prime}Q_{\prime\prime}} & \overline{Q_{\prime\prime}} \\ \multicolumn{2}{l}{X_{\prime} - X_{\prime\prime}Q_{\prime\prime} = -X_{\prime\prime\prime};} \end{array} \quad \begin{array}{r|l} X_{\prime\prime} & X_{\prime\prime\prime} \\ \underline{X_{\prime\prime\prime}Q_{\prime\prime\prime}} & \overline{Q_{\prime\prime\prime}} \\ \multicolumn{2}{l}{X_{\prime\prime} - X_{\prime\prime\prime}Q_{\prime\prime\prime} = -X_{\prime\prime\prime\prime}.} \end{array}$$

We thus obtain the series of quantities

$$X,\ X_{\prime},\ X_{\prime\prime},\ X_{\prime\prime\prime},\ X_{\prime\prime\prime\prime},\ \ldots\ldots\ X_m,$$

each of which is of a lower degree with respect to x than the preceding, and the last is altogether independent of x, that is, does not contain x.

We now substitute for x in the above functions any two numbers p and q, of which p is less than q. The substitution of p will give results either positive or negative. If we only take account of the *signs*, we shall obtain a certain number of *variations* and a certain number of *permanences*.

The substitution of q for x will give a second series of signs, presenting a certain number of variations and permanences. The following, then, is

THE THEOREM OF STURM.

The difference between the number of variations of the first row of signs and that of the second, is equal to the number of real roots of the given equation comprised between p *and* q.

(308.) In order to simplify the demonstration of this theorem, we shall premise two Lemmas; and for convenience we shall call X the *primitive* function, and $X_{\prime}$, $X_{\prime\prime}$, $X_{\prime\prime\prime}$, &c., *auxiliary* functions.

LEMMA 1. *Of the series* X, $X_{\prime}$, $X_{\prime\prime}$, &c., *two consecutive functions cannot both vanish for the same value of* x.

From the method in which $X_{\prime}$, $X_{\prime\prime}$, &c., are obtained, we have the following equations:

$$\begin{aligned}
X &= X_{,} Q_{,} - X_{,,} \quad (1).\\
X_{,} &= X_{,,} Q_{,,} - X_{,,,} \quad (2).\\
X_{,,} &= X_{,,,} Q_{,,,} - X_{,,,,} \quad (3).\\
&\text{“} \quad \text{“} \quad \text{“}\\
X_{m-2} &= X_{m-1} Q_{m-1} - X_{m} \quad (m-1).
\end{aligned}$$

Now suppose $X_{,,} = 0$, and $X_{,,,} = 0$; then, by equation (3), we have $X_{,,,,} = 0$. Hence, since $X_{,,,} = 0$, and $X_{,,,,} = 0$; therefore, by equation (4), we must have $X_{v} = 0$; and, proceeding in this manner, we shall find that $X_{m} = 0$, which is *absurd*, since it was shown, Art. 307, that this final remainder must be independent of x, and must therefore remain unchanged for every value of x.

LEMMA II. *If one of the auxiliary functions vanishes for any particular value of* x, *the two adjacent functions must have contrary signs for the same value of* x.

For, by equation (3), we have

$$X_{,,} = X_{,,,} Q_{,,,} - X_{,,,,};$$

and if $X_{,,,} = 0$, then $X_{,,} = -X_{,,,,}$; that is, $X_{,,}$ and $X_{,,,,}$ have contrary signs.

DEMONSTRATION OF THE THEOREM.

(309.) Suppose all the real roots of the equations

$$X = 0,\ X_{,} = 0,\ X_{,,} = 0,\ X_{,,,} = 0,\ \&c.,$$

to be arranged in a series in the order of magnitude, beginning with the least. Let p be less than the least of these roots, and let it increase continually until it becomes equal to q, which we suppose to be greater than the greatest of these roots. Now so long as p is *less* than any of the roots, no change of signs will occur from the substitution of p for x in any of these functions, Art. 304; but when p arrives at a root of any of the auxiliary equations, its substitution for x reduces that polynomial to zero, and neither the preceding nor succeeding function can vanish for the same value of x (Lemma I.), and these two adjacent functions have contrary signs (Lemma II.). Hence the entire number of variations of sign is not affected by the vanishing of any of the auxiliary functions; for the three adjacent functions must reduce to

$$+,\ 0,\ -,\ \text{or}\ -,\ 0,\ +.$$

Here is one variation, and there will also be one variation if we supply the place of the 0 with either + or —; thus,

$$+, +, -, \text{ or } -, +, +,$$
$$+, -, -, \text{ “ } -, -, +.$$

Suppose, now, p to pass from a number very little smaller, to a number very little greater than a root of the primitive equation

$$X=0,$$

the sign of X will be changed from + to —, or from — to +, Art. 304. The signs of X and $X_,$ constitute a variation before the change, and a permanence after the change. Hence the change of sign of the function X occasions a loss of one variation of sign.

Again, while p increases from a number very little smaller to a number very little greater than another root of $X=0$, a second variation will be changed into a permanence, and so on for the other roots of the primitive equation.

Hence the number of variations *lost*, when p increases from $-\infty$ to $+\infty$, must be equal to the whole number of real roots of the equation $X=0$.

(310.) EXAMPLES.

Ex. 1. How many real roots has the equation

$$x^3-6x^2+11x-6=0?$$

Here we have $X_, = 3x^2-12x+11.$

Dividing $x^3-6x^2+11x-6$ by $3x^2-12x+11$, as in the method for finding the greatest common divisor, Art. 250, we have for a remainder $-2x+4$. Hence, rejecting the factor 2, $X_{,,}=x-2$. Dividing $X_,$ by $X_{,,}$, we have for a remainder -1. Therefore, $X_{,,,}=+1$.

Hence we have

$$X \;=x^3-6x^2+11x-6.$$
$$X_, \;=3x^2-12x+11.$$
$$X_{,,} =x-2.$$
$$X_{,,,}=+1.$$

Now, if we substitute for x in these functions, the number given below, the signs of the results will be as follows:

Assumed Values of x.	Resulting Signs.	Variations.
$-\infty$	$- + - +$	giving 3 variations.
0	$- + - +$	" 3 "
$+.9$	$- + - +$	" 3 "
$+1$	$0 + - +$	" 2 "
$+1.1$	$+ + - +$	" 2 "
$+1.9$	$+ - - +$	" 2 "
$+2.$	$0 - 0 +$	" 1 "
$+2.1$	$- - + +$	" 1 "
$+2.9$	$- + + +$	" 1 "
$+3$	$0 + + +$	" 0 "
$+3.1$	$+ + + +$	" 0 "
$+\infty$	$+ + + +$	" 0 "

Here the three roots of this equation are seen to be 1, 2, 3, and no change of sign occurs by the substitution for x of any number less than 1; but when p exceeds 1, there is a change of sign in the original equation from $-$ to $+$, by which one variation is *lost.* When $p=2$, two of the functions disappear simultaneously, showing that 2 is a root of the second derived function as well as of the original equation, and a second variation of sign is *lost.* Also, when p becomes equal to 3, a third variation is lost; and there are no farther changes of sign arising from the substitution of any numbers between 3 and $+\infty$.

There are *three* changes of sign of the primitive function, *two* of the first auxiliary function, and *one* of the second auxiliary function; but no variation is lost by the change of sign of any of the auxiliary functions; while every change of sign of the primitive function occasions a loss of one variation.

Ex. 2. How many real roots has the equation

$$x^3-5x^2+8x-1=0?$$

Here we find

$$\begin{aligned}
X\ &= x^3-5x^2+8x-1.\\
X_{\prime}\ &= 3x^2-10x+8.\\
X_{\prime\prime} &= 2x-31.\\
X_{\prime\prime\prime} &= -2295.
\end{aligned}$$

When $x=-\infty$, the signs are $-+--$, giving 2 variations
$x=+\infty$, " $+++-$, " 1 "

Hence this equation has but one real root, and consequently must have two imaginary roots.

Ex. 3. How many real roots has the equation

$$x^4-2x^3-7x^2+10x+10=0\,?$$

Here we have

$X = x^4-2x^3-7x^2+10x+10.$
$X_{\prime} = 4x^3-6x^2-14x+10$; or $2x^3-3x^2-7x+5.$
$X_{\prime\prime} = 17x^2-23x-45.$
$X_{\prime\prime\prime} = 152x-305.$
$X_{\prime\prime\prime\prime} = +524785.$

When $x=-\infty$, the signs are $+-+-+$, giving 4 variations;
$x=+\infty$, " $+++++$, " 0 "

Hence the four roots of this equation are real.

If we try different values for x, we shall find that

When $x=-3$, the signs are $+-+-+$, giving 4 variations;
$x=-2$, " $-++-+$, " 3 "
$x=-1$, " $-++-+$, " 3 "
$x=\ \ 0$, " $++--+$, " 2 "
$x=+1$, " $+---+$, " 2 "
$x=+2$, " $+---+$, " 2 "
$x=+3$, " $+++++$, " 0 "

Hence this equation has two positive roots between 2 and 3; one negative root between 0 and -1; and one negative root between -2 and -3.

Ex. 4. How many real roots has the equation

$$x^3-7x+7=0\,?$$

Ans. Three: viz., two between 1 and 2, and one between -3 and -4.

Ex. 5. How many real roots has the equation

$$2x^4-13x^2+10x-19=0\,?$$

Ans. Two: one positive and one negative.

Ex. 6. How many real roots has the equation

$$x^5-36x^3+72x^2-37x+72=0\,?$$

Ans. Three.

SECTION XX.

SOLUTION OF NUMERICAL EQUATIONS.

(311.) We will first consider the method of finding the *integral* roots of an equation, and will begin with forming the equation whose roots are 2, 3, 4, and 5. This equation must consist of the factors

$$(x-2)\,(x-3)\,(x-4)\,(x-5)=0.$$

If we perform the multiplication (which is most expeditiously done by the method of detached coefficients shown in Art. 64), we obtain the equation

$$x^4-14x^3+71x^2-154x+120=0.$$

We know that this equation is divisible by $x-5$. Let us perform the division by the method of detached coefficients shown in Art. 80.

$$
\begin{array}{rrrrr|l}
\text{A} & \text{B} & \text{C} & \text{D} & \text{V} & \quad a \\
1 & -14 & +71 & -154 & +120 & 1-5=\text{divisor.} \\ \cline{6-6}
1 & -5 & & & & 1-9+26-24=\text{quotient.} \\ \cline{1-2}
 & -9 & +71 & & & \\
 & -9 & +45 & & & \\ \cline{2-3}
 & & +26 & -154 & & \\
 & & +26 & -130 & & \\ \cline{3-5}
 & & & -24 & +120 & \\
 & & & -24 & +120. &
\end{array}
$$

Supplying the powers of x, we obtain for a quotient

$$x^3-9x^2+26x-24=0.$$

This operation may be still farther abridged, as follows:

Represent the root 5 by a, and the coefficients of the given equation by A, B, C, D, ... V.

We first multiply a by A, and subtract the product from B; the remainder, -9, we multiply by a, and subtract the product from C; the remainder, $+26$, we multiply again by a, and subtract from D; the remainder, -24, we multiply by a, and, subtracting from V, nothing remains. If we take care to change the sign of a, we may substitute addition for subtraction in the above statement; and if we set down only the successive remainders, the work will be as follows:

$$\begin{array}{ccccc|c} A & B & C & D & V & a \\ 1 & -14 & +71 & -154 & +120 & \underline{5} \\ 1 & -9 & +26 & -24, & & \end{array}$$

and the rule will be,

Multiply A *by* a, *and add the product to* B; *set down the sum, multiply it by* a, *and add the product to* C; *set down the sum, multiply it by* a, *and add the product to* D, *and so on. If the equation is exactly divisible, the final product will be equal to the last term* V, *taken with a contrary sign.*

The coefficients above obtained are the coefficients of a cubic equation whose roots are 2, 3, 4. The equation may therefore be divided by $x-4$, and the operation will be as follows:

$$\begin{array}{l|l} 1-9+26-24 & 4 \\ 1-5+6. & \end{array}$$

These, again, are the coefficients of a quadratic equation whose roots are 2 and 3. Dividing again by $x-3$, we have

$$\begin{array}{l|l} 1-5+6 & 3 \\ 1-2, & \end{array}$$

which are the coefficients of the binomial factor $x-2$.

These three operations of division may be exhibited together as follows:

$$\begin{array}{l|l} 1-14+71-154+120 & 5,\text{ first divisor.} \\ 1-\ \ 9+26-\ \ 24 & 4,\text{ second divisor.} \\ 1-\ \ 5+6 & 3,\text{ third divisor.} \\ 1-\ \ 2. & \end{array}$$

(312.) The method here explained will enable us to find all the integral roots of an equation. For this purpose, we make trial of different numbers in succession, all of which must be divisors of the last term of the equation. If any division leaves a remainder, we reject this divisor; if the division leaves no remainder, the divisor employed is a *root* of the equation. Thus, by a few trials, all the integral roots may be easily found.

Ex. 2. Find the seven roots of the equation

$$x^7+x^6-14x^5-14x^4+49x^3+49x^2-36x-36=0.$$

We take the coefficients separately, as in the last example, and try in succession all the divisors of 36, both positive and negative, rejecting such as leave a remainder. The operation is as follows:

$1+1-14-14+49+49-36-36$	1, first divisor.
$1+2-12-26+23+72+36$	2, second divisor.
$1+4-4-34-45-18$	3, third divisor.
$1+7+17+17+6$	-1, fourth divisor.
$1+6+11+6$	-1, fifth divisor.
$1+5+6$	-2, sixth divisor.
$1+3$	-3, seventh divisor.

Hence the seven roots are,

$$1,\ 2,\ 3,\ -1,\ -1,\ -2,\ -3.$$

Ex. 3. Find the six roots of the equation

$$x^6+5x^5-81x^4-85x^3+964x^2+780x-1584=0.$$

$1+5-81-85+964+780-1584$	1.
$1+6-75-160+804+1584$	4.
$1+10-35-300-396$	6.
$1+16+61+66$	-2.
$1+14+33$	-3.
$1+11$	-11.

The six roots, therefore, are,

$$1,\ 4,\ 6,\ -2,\ -3,\ -11.$$

Ex. 4. Find the five roots of the equation

$$x^5+6x^4-10x^3-112x^2-207x-110=0.$$

$$\begin{array}{l|l}
1+6-10-112-207-110 & -1. \\
1+5-15-\ \ 97-110 & -2. \\
1+3-21-\ \ 55 & -5. \\
1-2-11 &
\end{array}$$

Three of the roots, therefore, are

$$-1,\ -2,\ -5.$$

The two remaining roots may be found by the ordinary method of quadratic equations. Supplying the letters to the last coefficients, we have

$$x^2-2x-11=0.$$

Hence $$x=1\pm\sqrt{12}.$$

Ex. 5. Find the four roots of the equation

$$x^4+2x^3-7x^2-8x+12=0.$$

Ex. 6. Find the four roots of the equation

$$x^4-55x^2-30x+504=0.$$

Ex. 7. Find all the roots of the equation

$$x^4-25x^2+60x-36=0.$$

Ex. 8. Find all the roots of the equation

$$x^5+5x^4+x^3-16x^2-20x-16=0.$$

HORNER'S METHOD.

(313.) The preceding method furnishes the roots of an equation only when they are expressed by *whole numbers.* When the roots are incommensurable, we employ the following method, which is substantially the same as published by Horner in 1819.

The Theorem of Sturm, together with Art. 304, enables us to find the *integral part* of any real root of the equation proposed. We then transform the equation into another having its roots less than those of the preceding by the number just found, Art. 302. We discover again, by Art. 304, the first figure of the root of this equation, which will be the first decimal figure of the root of the original equation. Again, we transform the last equation into another having its roots less than those of the preceding by this decimal figure. We thus discover the second decimal figure of the root; and proceeding in this manner from one transformation to another, we are enabled to discover the successive figures of the root, and may carry the approximation to any degree of accuracy required.

Example 1. Find the roots of the cubic equation

$$x^3 + 3x^2 + 5x = 178.$$

The several functions given by Sturm's Theorem are,

$$\begin{aligned} X &= x^3 + 3x^2 + 5x - 178, \\ X_{\prime} &= 3x^2 + 6x + 5, \\ X_{\prime\prime} &= -4x + 539, \\ X_{\prime\prime\prime} &= -884579. \end{aligned}$$

Now for $x = +\infty$, the signs are, $+\,+\,-\,-$, giving 1 variation.
$x = -\infty$, " $-\,+\,+\,-$, " 2 "

Therefore two values of x are *imaginary*, and the equation has but one real root.

We readily find, by the method of Art. 310, that this root lies between 4 and 5. The first figure of the root, therefore, is 4. To ascertain the second figure, we transform the given equation into another in which the value of x is diminished by 4, which is done by substituting for x, $y + 4$. We thus obtain

$$y^3 + 15y^2 + 77y = 46.$$

The first figure of the root of this equation, according to Art. 304, is .5. Now transform the last equation into another in which the value of y is diminished by .5, which is done by substituting for y, $z + .5$. We thus obtain

$$z^3 + 16.5z^2 + 92.75z = 3.625.$$

The first figure of the root of this equation is .03. We must now transform this equation into another in which the value of z is diminished by .03, which is done by substituting for z, $v + .03$. We thus obtain

$$v^3 + 16.59v^2 + 93.7427v = .827623.$$

The first figure of the root of this equation is .008.

In order to find the next figure, we must transform the last equation into another in which the value of v is diminished by .008, and so on.

(314.) This method would be very laborious if we were obliged to deduce the successive equations from each other by the ordinary method of substitution; but they may all be derived from each other by a very simple law. Thus, let

$$Ax^3 + Bx^2 + Cx = D \quad (1)$$

be any cubic equation; and let the first figure of its root be denoted by a, the second by a', the third by a'', and so on.

If we substitute a for x in equation (1), we shall have

$$Aa^3 + Ba^2 + Ca = D.$$

Whence $$a = \frac{D}{C + Ba + Aa^2} \quad (2).$$

If we put y for the sum of all the figures of the root except the first, we shall have $x = a + y$; and substituting this value for x in equation (1), we obtain

$$\left.\begin{array}{r} Aa^3 + 3Aa^2y + 3Aay^2 + Ay^3 \\ + \ Ba^2 \ + 2Bay \ + By^2 \\ + \ Ca \ + Cy \end{array}\right\} = D;$$

or, arranging according to the powers of y,

$$Ay^3 + (B + 3Aa)y^2 + (C + 2Ba + 3Aa^2)y = D - Ca - Ba^2 - Aa^3 \quad (3).$$

Let us put B' for the coefficient of y^2, C' for the coefficient of y, and D' for the right member of the equation, and we have

$$Ay^3 + B'y^2 + C'y = D' \quad (4).$$

This equation is of the same form as equation (1); and proceeding in the same manner, we shall find

$$a' = \frac{D'}{C' + B'a' + Aa'^2} \quad (5),$$

where a' is the first figure of the root of equation (4), or the second figure of the root of equation (1).

Putting z for the sum of all the remaining figures, we have $y = a' + z$; and substituting this value in equation (4), we shall obtain a new equation of the same form, which may be written

$$Az^3 + B''z^2 + C'z = D'' \quad (6);$$

and in the same manner we might proceed with the remaining figures.

Equation (2) furnishes the value of the first figure of the root; equation (5) the second figure, and similar equations would furnish the remaining figures. Each of these expressions involves the unknown quantity which is sought, and might therefore appear to be useless in practice. When, however, the root has been already found to several decimal places, the value of the terms Ba and Aa^2 will be very small compared with C, and a will be very nearly equal to $\frac{\text{D}}{\text{C}}$. C, then, may in all cases be employed as an *approximate divisor*, which will probably furnish a new figure of the root. Thus, in the above example, all the figures of the root after the first, are found by division.

$$46 \div 77 = .5.$$

$$3.625 \div 92.75 = .03.$$

$$.827 \div 93.74 = .008.$$

If we multiply the first coefficient A by a, the first figure of the root, and add the product to the second coefficient, we shall have

$$\text{B} + \text{A}a \quad (7).$$

If we multiply this expression by a, and add the product to the third coefficient, we shall have

$$\text{C} + \text{B}a + \text{A}a^2 \quad (8).$$

If we multiply this expression by a, and subtract the product from D, we shall have

$$\text{D} - \text{C}a - \text{B}a^2 - \text{A}a^3,$$

which is the quantity represented by D′ *in equation* (4).

Again, multiplying the first coefficient by a, and adding the product to expression (7), we obtain

$$\text{B} + 2\text{A}a \quad (9).$$

Multiplying this expression by a, and adding the product to expression (8), we have

$$C + 2Ba + 3Aa^2,$$

which is the coefficient of y *in equation* (4).

Again, multiplying the first coefficient by a, and adding the product to expression (9), we have

$$B + 3Aa,$$

which is the coefficient of y^2 *in equation* (4),

We have thus obtained the coefficients of the first transformed equation; and by operating in the same manner upon these coefficients, we shall obtain the coefficients of the second transformed equation, and so on; and the successive figures of the root are found by dividing D by C, D′ by C′, D″ by C″, and so on.

(315.) The preceding method is summed up in the following

RULE.

Represent the coefficients of the different terms by A, B, C, *and the right-hand member of the equation by* D. *Having found* a, *the first figure of the root, multiply* A *by* a, *and add the product to* B. *Multiply this sum by* a, *and add the product to* C. *Multiply this last sum by* a, *and subtract the product from* D; *the remainder will be the* FIRST DIVIDEND.

Again, multiply A *by* a, *and add the product to the last number under* B. *Multiply this sum by* a, *and add the product to the last number under* C; *this last sum will be the* FIRST DIVISOR.

Again, multiply A *by* a, *and add the product to the last number under* B.

Find the second figure of the root by dividing the first dividend by the first divisor, and proceed with this second figure precisely as was done with the first figure.

The second figure of the root obtained by division, will frequently furnish a result too large to be subtracted from the remainder D′, in which case we must assume a different figure. After the second figure of the root has been obtained, there will seldom be any farther uncertainty of this kind.

The operation for finding a root of the equation

$$x^3 + 3x^2 + 5x = 178$$

will then proceed as follows:

A	B	C	D	a
1	+3	+5	=178	(4.5388 = x.
	4	28	132	
	7	33	46 = 1st dividend.	
	4	44	42.375	
	11	77 = 1st divisor.	3.625 = 2d dividend.	
	4	7.75	2.797377	
	15.5	84.75	.827623 = 3d dividend.	
	.5	8.00	.751003872	
	16.0	92.75 = 2d divisor.	.076619128 = 4th dividend.	
	.5	.4959		
	16.53	93.2459		
	3	.4968		
	16.56	93.7427 = 3d divisor.		
	3	.132784		
	16.598	93.875484		
	8	.132848		
	16.606	94.008332 = 4th divisor.		

Having found one root, we may depress the equation

$$x^3 + 3x^2 + 5x - 178 = 0$$

to a quadratic, by dividing it by $x - 4.5388$. We thus obtain

$$x^2 + 7.5388x + 39.2173 = 0,$$

where x is evidently imaginary, because q is negative and greater than $\frac{p^2}{4}$. See Art. 195.

After thus obtaining the root to five or six decimal places, *several more* figures will be correctly obtained by simply dividing the last dividend by the last divisor.

Example 2. Find the roots of the equation

$$x^3 + 11x^2 - 102x = -181.$$

The first figure of one of the roots we readily find to be 3. We then proceed, according to the Rule, to obtain the root to four decimal places, after which two more will be obtained correctly by division.

A	B	C	D	a
1	+11	—102	= —181	(3.21312 = x.
	3	42	—180	
	14	—60	—1 = 1st dividend.	
	3	51	—.992	
	17	—9 = 1st divisor.	—.008 = 2d dividend.	
	3	4.04	—.006739	
	20.2	—4.96	—.001261 = 3d dividend.	
	2	4.08	—.001217403	
	20.4	—0.88 = 2d divisor.	—.000043597 = 4th dividend.	
	2	.2061		
	20.61	—.6739		
	1	.2062		
	20.62	—.4677 = 3d divisor.		
	1	.061899		
	20.633	—.405801		
	3	.061908		
	20.636	—.343893 = 4th divisor.		

The two remaining roots may be found in the same way, or by depressing the original equation to a quadratic. Those roots are,

$$3.22952$$
$$-17.44265.$$

When a power of x is wanting in the proposed equation, we must supply its place with a cipher.

Ex. 3. Find all the roots of the cubic equation

$$x^3-7x=-7$$

The work of the following example is exhibited in an abbreviated form. Thus, when we multiply A by a, and add the product to B, we set down simply this *result.* We do the same in the next column, thus dispensing with half the number of lines employed in the preceding example. Moreover, we may omit the ciphers on the left of the successive dividends, if we pay proper attention to the local value of the figures. Thus, it will be seen that in the operation for finding each successive figure of the root, the decimals under B increase *one* place, those under C increase *two* places, and those under D increase *three* places.

1 +0	—7	=—7 (1.356895867 = x.
1	—6	—6
2	—4 = 1st div'r.	—1 = 1st dividend.
3.3	—3.01	—.903
3.6	—1.93 = 2d div'r.	—97 = 2d dividend.
3.95	—1.7325	86625
4.00	—1.5325 = 3d div'r.	10375 = 3d dividend.
4.056	—1.508164	9048984
4.062	—1.483792 = 4th div'r.	1326016 = 4th dividend.
4.0688	—1.48053696	1184429568
4.0696	—1.47728128 = 5th div'r.	141586432 = 5th div'd.
4.07049	—1.4769149359	132922344231
4.07058	—1.4765485837 = 6th div'r.	8664087769 = 6th div'd.

Having proceeded thus far, four more figures of the root, 5867, are found by dividing the sixth dividend by the sixth divisor.

We may find the two remaining roots by the same process; or, after having obtained one root, we may depress the equation

$$x^3 - 7x + 7 = 0$$

to a quadratic equation, by dividing by $x - 1.356895867$, and we shall obtain

$$x^2 + 1.356895867x - 5.158833606 = 0.$$

Solving this equation, we obtain

$$x = - .678447933 \pm \sqrt{5.619125204}$$
$$= - 3.048917338, \text{ or } + 1.692021471,$$

the two remaining roots.

Ex. 4. Find a root of the equation

$$2x^3 + 3x^2 = 850.$$

2	3	0	=850 (7.0502562208
	17	119	833
	31	336=1st divisor.	17=1st dividend.
	45.10	338.2550	16.912750
	45.20	340.5150=2d divisor.	87250=2d dividend.
	45.3004	340.52406008	68104812016
	45.3008	340.53312024=3d div.	19145187984=3d div'd.
	45.30130	340.5353853050	17026769265250
	45.30140	340.5376503750=4th div.	2118418718750=4th div.

Dividing the fourth dividend by the fourth divisor, we obtain the figures 62208, which make the root correct to the tenth decimal place.

The two remaining values of x may be easily shown to be imaginary.

When a negative root is to be found, we change the signs of the alternate terms of the equation, Art. 299, and proceed as for a positive root.

Ex. 5. Find a root of the equation

$$5x^3 - 6x^2 + 3x = -85.$$

Changing the signs of the alternate terms, it becomes

$$5x^3 + 6x^2 + 3x = +85.$$

5	+6	+3	+85 (2.16139.
	16	35	70
	26	87 = 1st divisor.	15 = 1st dividend.
	36.5	90.65	9.065
	37.0	94.35 = 2d divisor.	5.935 = 2d dividend.
	37.80	96.6180	5.797080
	38.10	98.9040 = 3d divisor.	137920 = 3d dividend.
	38.405	98.942405	98942405
	38.410	98.980815 = 4th div'r.	38977595 = 4th div'd.
	38.4165	98.99233995	29697701985
	38.4180	99.00386535 = 5th div'r.	9279893015 = 5th div'd.

Hence one root of the equation

$$5x^3 - 6x^2 + 3x = -85$$

is -2.16139.

The same method is applicable to the extraction of the cube root of numbers.

Ex. 6. Let it be required to extract the cube root of 9; in other words, it is required to find a root of the equation

$$x^3 = 9.$$

1	0	0	9 (2.0800838.
	2	4	8
	4	12=1st divisor.	1=1st dividend.
	6.08	12.4864	.998912
	6.16	12.9792=2d divisor.	1088=2d dividend.
	6.24008	12.9796992064	1038375936512
	6.24016	12.9801984192=3d div.	49624063488=3d div.
	6.240243	12.980217139929	38940651419787
	6.240246	12.980235860667=4th d.	10683412068213=4th d.

Ex. 7. Find one root of the equation

$$x^3 + x^2 = 500.$$

Ans. $x = 7.617280$.

Ex. 8. Find one root of the equation

$$x^3 + x^2 + x = 100.$$

Ans. $x = 4.264430$.

Ex. 9. Find one root of the equation

$$2x^3 + 3x^2 - 4x = 10.$$

Ans. $x = 1.624819$.

Ex. 10. Extract the cube root of 48228544.

Ans. 364

Ex. 11. There are two numbers whose difference is 2, and whose product, multiplied by their sum, makes 120. What are those numbers?

Ex. 12. It is required to find two numbers whose difference is 6, and such that their sum, multiplied by the difference of their cubes, may produce 5040.

Ex. 13. Required two numbers whose difference is 5, and such, that, if the less be multiplied by the square root of the greater, the product may be 12.

Ex. 14. There are two numbers whose difference is 4; and the product of this difference, by the sum of their cubes, is 3416. What are the numbers?

Ex. 15. Several persons form a partnership, and establish a certain capital, to which each contributes ten times as many dollars as there are persons in company. They gain 6 *plus* the number of partners per cent., and the whole profit is $392. How many partners were there?

Ex. 16. A company of merchants have a common stock of $4775, and each contributes to it twenty-five times as many dollars as there are partners, with which they gain as much per cent as there are partners. Now on dividing the profit, it is found, after each has received six times as many dollars as there are persons in the company, that there still remains $126. Required the number of merchants?

Ans. 7, 8, or 9.

EQUATIONS OF THE FOURTH AND HIGHER DEGREES.

(316.) The method already explained for cubic equations is applicable to equations of every degree. For the fourth degree, we shall have one more column of products, but the operations are all conducted in the same manner, as will be seen from the following example.

Ex. 1. Find the four roots of the equation

$$x^4 - 8x^3 + 14x^2 + 4x = 8.$$

By Sturm's Theorem, we find that these roots are all real; three positive, and one negative.

We then proceed as follows:

1	—8	+14	+ 4	=8 (5.2360679.
	—3	— 1	— 1	—5
	+2	+ 9	+44=1st divisor.	13=1st dividend.
	7	44	53.288	10.6576
	12.2	46.44	63.072=2d div'r.	2.3424=2d dividend.
	12.4	48.92	64.626747	1.93880241
	12.6	51.44	66.193068=3d d.	.40359759=3d div'd.
	12.83	51.8249	66.509117736	.399054706416
	12.86	52.2107	66.825633024=4th d.	4542883584=4th d.
	12.89	52.5974		
	12.926	52.674956		
	12.932	52.752548		

and by division we obtain the four figures 0679.

The other three roots are, $\begin{cases} x = 2.7320508, \\ x = .7639320, \\ x = -.7320508. \end{cases}$

Ex. 2. Find the roots of the equation

$$x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 20.$$

Here we have, according to Sturm's Theorem,

$$\begin{aligned} X &= x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 20, \\ X_{,} &= 5x^4 + 8x^3 + 9x^2 + 8x + 5, \\ X_{,,} &= -7x^3 - 21x^2 - 42x + 255, \\ X_{,,,} &= -13x + 14, \\ X_{,,,,} &= -. \end{aligned}$$

When $x = +\infty$, the signs are + + — — —, one variation.

$x = -\infty$, " — + + + —, two "

Hence the equation has one real and four imaginary roots.

We then proceed as follows:

1	+2	+3	+4	+5	+20 (1.125789.
	3	6	10	15	15
	4	10	20	35 = 1st divisor.	5 = 1st dividend.
	5	15	35	38.7171	3.87171
	6	21	37.171	42.6585 = 2d divisor.	1.12829 = 2d dividend.
	7.1	21.71	39.414	43.5027 2016	.87005 44032
	7.2	22.43	41.730	44.3566 2080 = 3d divisor.	.25823 55968 = 3d dividend.
	7.3	23.16	42.211 008	44.5731 44750625	.22286 5723753125
	7.4	23.90	42.695 032	44.7902 83203125 = 4th div.	3536 9873046875 = 4th div.
	7.5 2	24.05 04	43.182 080		
	7.5 4	24.20 12	43.304 790125		
	7.5 6	24.35 24	43.427 690500		
	7.5 8	24.50 40			
	7.6 05	24.54 2025			
	7.6 10	24.58 0075			

Dividing the fourth dividend by the fourth divisor, we obtain the figures 789.

When we wish to obtain a root correct to a limited number of places, we may save much of the labor of the operation by cutting off all figures beyond a certain decimal. Thus if, in the example above, we cut off all beyond five decimal places in the successive dividends, and all beyond four decimal places in the divisors, it will not affect the first six decimal places in the root.

Ex. 3. Find the roots of the equation

$$x^4 - 12x^2 + 12x = 3.$$

$$\text{Ans.} \left\{ \begin{array}{l} x = +\ 2.858083, \\ x = +\ \ .606018, \\ x = +\ \ .443277, \\ x = -\,3.907378. \end{array} \right.$$

Ex. 4. Find the roots of the equation

$$x^4 - 8x^3 + 14x^2 + 4x = 8.$$

$$\text{Ans.} \left\{ \begin{array}{l} x = +\ 5.236068, \\ x = +\ \ .763932, \\ x = +\ 2.732050, \\ x = -\ \ .732050. \end{array} \right.$$

Ex. 5. Find the roots of the equation

$$x^5 - 10x^3 + 6x = -1.$$

$$\text{Ans.} \left\{ \begin{array}{l} x = -\,3.065315, \\ x = -\ \ .691576, \\ x = -\ \ .175675, \\ x = +\ \ .879508, \\ x = +\ 3.053058. \end{array} \right.$$

Ex. 6. Find one root of the equation

$$x^5 + 3x^4 + 2x^3 - 3x^2 - 2x = 2.$$

Ans. $x = 1.059109$.

Ex. 7. Required the fourth root of 18339659776.

Ans. 368.

Ex. 8. Required the fifth root of 26287667869473.

Ans. 483.

RESOLUTION OF EQUATIONS BY APPROXIMATION.

(317.) The method of Horner for finding the incommensurable roots of a numerical equation is generally better than any other; nevertheless, the method by approximation may sometimes be preferred. We shall explain the method of Newton, and that of Double Position.

METHOD OF NEWTON.

This method supposes that we have already determined nearly the value of one root; that we know, for example, that such a value exceeds 4, and that it is less than 5. In this case, if we suppose the exact value $=4+y$, we are certain that y expresses a proper fraction. Now as y is less than unity, the square of y, its cube, and, in general, all its higher powers, will be much less with respect to unity; and for this reason, since we only require an approximation, they may be neglected in the calculation. When we have nearly determined the fraction y, we shall know more exactly the root $4+y$; from which we proceed to determine a new value still more exact, and we may continue the approximation as far as we please.

We will illustrate this method by an easy example, requiring by approximation the root of the equation

$$x^2 = 20.$$

Here we perceive that x is greater than 4, and less than 5. If we suppose $x = 4 + y$, we shall have

$$x^2 = 16 + 8y + y^2 = 20.$$

But, as y^2 must be quite small, we shall neglect it, and we have

$$16 + 8y = 20, \text{ or } 8y = 4.$$

Whence $y = .5$, and $x = 4.5$, which already approaches near the true root. If we now suppose $x = 4.5 + z$, we are sure that z expresses a fraction much smaller than y, and that we may neglect z^2 with greater propriety. We have, therefore,

$$x^2 = 20.25 + 9z = 20, \text{ or } 9z = -.25.$$

Consequently, $z = -.0278.$

Therefore, $x = 4.5 - .0278 = 4.4722.$

If we wish to approximate still nearer to the true value, we must make $x = 4.4722 + v$, and we should have

$$x^2 = 20.00057284 + 8.9444v = 20.$$

So that $$8.9444v = -.00057284.$$

Whence $$v = -.0000640.$$

Therefore, $x = 4.4722 - .0000640 = 4.4721360$,

a value which is correct to the last decimal place.

(318.) In order to generalize this method, suppose the given equation to be

$$x^2 = \text{D},$$

and that we have found x to be greater than a, but less than $a + 1$. If we make $x = a + y$, y must be less than unity, and y^2 may be neglected, as a very small quantity; so that we shall have

$$x^2 = a^2 + 2ay = \text{D},$$

or $$2ay = \text{D} - a^2\text{; and } y = \frac{\text{D} - a^2.}{2a.}$$

Consequently, $$x = a + \frac{\text{D} - a^2}{2a} = \frac{a^2 + \text{D}}{2a}.$$

Now, if a was near the true value of x, this new expression $\frac{a^2 + \text{D}}{2a}$ will approach nearer; and, by substituting it for a, we shall obtain a new value, which may again be employed in order to approximate still nearer, and the same operation may be continued as long as we please.

This method of approximation may be employed in finding the roots of *all* equations.

Suppose we have the general equation of the third degree,

$$\text{A}x^3 + \text{B}x^2 + \text{C}x = \text{D},$$

in which a is very nearly the value of one of the roots. Let us make $x = a + y$; and, since y is less than unity, neglecting the powers above the first degree, we shall have

$$x^2 = a^2 + 2ay,$$
$$x^3 = a^3 + 3a^2y,$$

whence we have the equation

$$Aa^3 + 3Aa^2y + Ba^2 + 2Bay + Ca + Cy = D.$$

Whence $$y = \frac{D - Ca - Ba^2 - Aa^3}{C + 2Ba + 3Aa^2}.$$

The numerator of this fraction is the quantity represented by D′ on page 306; that is, it is the *dividend* in Horner's method, and the denominator is the *divisor;* and we have already seen that this quotient furnishes one or more figures of the root.

EXAMPLES.

Ex. 1. For an application of this method, take the equation

$$x^3 + 2x^2 + 3x = 50.$$

Here $A = 1$, $B = 2$, $C = 3$, and $D = 50$.

Therefore, $$y = \frac{50 - 3a - 2a^2 - a^3}{3 + 4a + 3a^2}.$$

We see that x is nearly equal to 3. If we substitute 3 for a, we shall have

$$y = -\frac{2}{21}.$$

Whence $x = 2.9$ nearly.

And if we substitute this new value instead of a, we shall find another still more exact.

Ex. 2. For another example, take the equation

$$x^5 - 6x = 10.$$

If we make $x = a + y$, we shall have

$$x^5 = a^5 + 5a^4y,$$

neglecting the higher powers of y, and, consequently,

$$a^5 + 5a^4y - 6a - 6y = 10.$$

Therefore, $$y = \frac{10 + 6a - a^5}{5a^4 - 6}.$$

Assume $a = 2$, and we obtain

$$y = \frac{10 + 12 - 32}{80 - 6} = -\frac{5}{37}.$$

Hence $x = 1.86$ nearly.

If we assume $a = 1.86$, we have

$$y = \frac{10 + 11.16 - 22.262}{59.844 - 6} = - .021.$$

Hence $x = 1.839$ nearly.

If we assume $a = 1.839$, we shall have

$$y = \frac{10 + 11.034 - 21.033352}{57.18694 - 6} = .00001266.$$

Therefore, $x = 1.83901266$.

Ex. 3. Given $x^3 - 9x = 10$, to find one value of x by approximation.

Ans. $x = 3.4494897$

Ex. 4. Given $x^3 + 9x^2 + 4x = 80$, to find one value of x by approximation.

Ans. $x = 2.4721359$.

METHOD OF DOUBLE POSITION.

(319.) Another method of finding the roots of an equation is by the rule of Double Position.

Substitute in the given equation two numbers as near the true root as possible, and observe the separate results. Then state the following proportion:

As the difference of these results,
Is to the difference of the two assumed numbers,
So is the error of either result,
To the correction required in the corresponding assumed number.

This being added to the number when too small, or subtracted from it when too great, will give the true root *nearly*. The number thus found, combined with any other that may be supposed to approach still nearer to the true root, may be assumed for another operation, which may be repeated till the root is determined to any degree of accuracy required.

EXAMPLES.

Ex. 1. Given $x^3 + x^2 + x = 100$, to find an approximate value of x.

Having ascertained, by a few trials, that x is more than 4, and less than 5, let us substitute these two numbers in the given equation, and calculate the results.

$$\text{By the first supposition,}\left\{\begin{aligned} x &= 4 \\ x^2 &= 16 \\ x^3 &= 64 \end{aligned}\right. \qquad \text{By the second supposition,}\left\{\begin{aligned} x &= 5 \\ x^2 &= 25 \\ x^3 &= 125 \end{aligned}\right.$$

$$\text{Result, } 84 \qquad\qquad \text{Result, } 155$$

Then $155 - 84 : 5 - 4 :: 100 - 84 : .22$.

Therefore, $4 + .22$, or 4.22, approximates nearly to the true root.

If, now, 4.2 and 4.3 be taken as the assumed numbers, and substituted in the given equation, we shall obtain the value of $x = 4.264$ nearly.

Again, assuming 4.264 and 4.265, and proceeding in the same manner, we shall find $x = 4.2644299$ very nearly.

This rule is founded on the supposition that the differences in the results are proportioned to the differences in the assumed numbers. This supposition is not strictly correct; but if we employ numbers near the true values, the error commonly is not very great, and it becomes less and less the farther we carry the approximation.

Ex. 2. Given $x^3 + 2x^2 - 23x - 70 = 0$, to find one value of x.

Ans. $x = 5.13450$.

Ex. 3. Given $x^3 - 15x^2 + 63x - 50 = 0$, to find one value of x.

Ans. $x = 1.028039$.

Ex. 4. Given $x^4 - 3x^2 - 75x = 10000$, to find one value of x.

Ans. $x = 10.2615$.

Ex. 5. Given $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321$, to find one value of x.

Ans. $x = 8.4144$

(320.) We will conclude this Section by finding *the different roots of unity.*

Ex. 1. Find the two roots of the equation $x^2 = 1$, or the square roots of unity.

Extracting the square root, we find

$$x = +1, \text{ or } -1.$$

Ex. 2. Find the three roots of the equation $x^3 = 1$, or the cube roots of unity.

Since one root of this equation is $x = 1$, the equation $x^3 - 1 = 0$ must be divisible by $x - 1$; and dividing, we obtain

$$x^2 + x + 1 = 0;$$

whence $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$, or $\dfrac{-1 \pm \sqrt{-3}}{2}$.

Hence the required roots are,

$$+1, \quad \frac{-1 + \sqrt{-3}}{2}, \quad \frac{-1 - \sqrt{-3}}{2}$$

which are the cube roots of unity.

These results may be easily verified. We have seen, on page 282, that the cube of $-1 \pm \sqrt{-3}$ is 8, which, divided by 8, the cube of the denominator, gives $+1$, as required.

Ex. 3. Find the four roots of the equation $x^4 = 1$, or the fourth roots of unity.

The square root of this equation is

$$x^2 = +1, \text{ or } = -1.$$

Hence the required roots are,

$$+1, \quad -1, \quad +\sqrt{-1}, \quad -\sqrt{-1}.$$

Ex. 4. Find the five roots of the equation $x^5 = 1$.

Since one root of this equation is $x = 1$, the equation $x^5 - 1$ must be divisible by $x - 1$; and dividing, we obtain

$$x^4 + x^3 + x^2 + x + 1 = 0.$$

Dividing, again, by x^2, we have

$$x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0 \quad (1).$$

Now put $$z = x + \frac{1}{x}.$$

Whence $$z^2 = x^2 + 2 + \frac{1}{x^2},$$

which, being substituted in equation (1), gives

$$z^2 + z - 1 = 0.$$

This equation, solved by the usual method, gives

$$z = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}, \text{ or } z = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}.$$

The values of x, deduced from the equation

$$z = x + \frac{1}{x},$$

or $$x^2 - zx = -1,$$

are,

$$x = \frac{z}{2} + \sqrt{\frac{z^2 - 4}{4}}, \text{ and } x = \frac{z}{2} - \sqrt{\frac{z^2 - 4}{4}},$$

in which the value of z being substituted, gives

$$x = \tfrac{1}{4}\left[\sqrt{5} - 1 \pm \sqrt{-10 - 2\sqrt{5}}\right],$$

or $$= -\tfrac{1}{4}\left[\sqrt{5} + 1 \mp \sqrt{-10 + 2\sqrt{5}}\right].$$

Hence the five fifth roots of unity are,

$$1$$

$$\tfrac{1}{4}\left[\sqrt{5} - 1 + \sqrt{-10 - 2\sqrt{5}}\right].$$

$$\tfrac{1}{4}\left[\sqrt{5} - 1 - \sqrt{-10 - 2\sqrt{5}}\right].$$

$$-\tfrac{1}{4}\left[\sqrt{5} + 1 - \sqrt{-10 + 2\sqrt{5}}\right].$$

$$-\tfrac{1}{4}\left[\sqrt{5} + 1 + \sqrt{-10 + 2\sqrt{5}}\right].$$

Ex. 5. Find the six roots of the equation $x^6 = 1$.

These are found by taking the square roots of the cube roots. Hence we have,

$$+1, \ -1, \ \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3}, \ -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3}.$$

Thus we see that unity has *two* square roots, *three* cube roots, *four* fourth roots, *five* fifth roots, *six* sixth roots, and, generally, n nth roots; or the nth root of unity admits of n different algebraic values. As, however, most of these roots are imaginary, they cannot be found by Horner's Method.

SECTION XXI.

LOGARITHMS.

(321.) IN a system of logarithms, all numbers are considered as the powers of some one number, arbitrarily assumed, which is called the *base* of the system; and *the exponent of that power of the base which is equal to any given number is called the logarithm of that number.*

Thus, if a be the base of a system of logarithms, and $a^2 = N$, then 2 is the logarithm of N; that is, 2 is the exponent of the power to which the base (a) must be raised to equal N.

If $a^3 = N'$, then 3 is the logarithm of N′ for the same reason;

And if $a^x = N''$, then x is called the logarithm of N″ in the system whose base is a.

The base of the common system of logarithms (called, from their inventor, Briggs' Logarithms) is the number 10. Hence all numbers are to be regarded as powers of 10. Thus, since

$10^0 = 1$,	0	is the logarithm of	1	in Briggs' system.
$10^1 = 10$,	1	"	10	"
$10^2 = 100$,	2	"	100	"
$10^3 = 1000$,	3	"	1000	"
$10^4 = 10000$,	4	"	10000	"
	&c.,		&c.,	&c.

From this it appears that, in Briggs' system, the logarithm of every number between 1 and 10 is some number between 0 and 1, *i. e.*, is a proper fraction. The logarithm of every number between 10 and 100 is some number between 1 and 2, *i. e.*, is 1 plus a fraction. The logarithm of every number between 100 and

1000 is some number between 2 and 3, *i. e.*, is 2 plus a fraction, and so on.

(322.) The preceding principles may be extended to fractions by means of negative exponents. Thus,

$\frac{1}{10}$ or $10^{-1}=0.1$; therefore, -1 is the logarithm of .1 in Briggs' system.
$\frac{1}{100}$ or $10^{-2}=0.01$; " -2 " .01
$\frac{1}{1000}$ or $10^{-3}=0.001$; " -3 " .001
$\frac{1}{10000}$ or $10^{-4}=0.0001$; " -4 " .0001

Hence it appears that the logarithm of every number between 1 and .1 is some number between 0 and -1, or may be represented by $-1+$ a fraction; the logarithm of every number between .1 and .01 is some number between -1 and -2, or may be represented by $-2+$ a fraction; the logarithm of every number between .01 and .001 is some number between -2 and -3, or is equal to $-3+$ a fraction, and so on.

The logarithms of most numbers, therefore, consist of an integer and a fraction. The integral part is called the *characteristic* or *index*, and may always be known from the following

RULE.

The index of the logarithm of any number greater than unity, is one less than the number of integral figures in the given number.

Thus, the logarithm of 297 is $2+$ a fraction; that is, the *index* of the logarithm of 297 is 2, which is one less than the number of integral figures. The index of the logarithm of 5673 is 3; of 73254, is 4, &c.

The index of the logarithm of a decimal fraction is a negative number, and is equal to the number of places by which its first significant figure is removed from the place of units.

Thus, the logarithm of .0046 is $-3+$ a fraction; that is, the index of the logarithm is -3, the first significant figure, 4, being removed three places from units.

GENERAL PROPERTIES OF LOGARITHMS.

(323.) Let N and N′ be any two numbers, x and x' their respective logarithms, a the base of the system. Then, by the definition, Art. 321,

$$N = a^x \quad (1),$$
$$N' = a^{x^1} \quad (2).$$

Multiplying together equations (1) and (2), we obtain

$$NN' = a^x a^{x^1}$$
$$= a^{x+x^1}.$$

Therefore, according to the definition of logarithms, $x + x'$ is the logarithm of NN′, since $x + x'$ is the index of that power of the base a which is equal to NN′; that is to say,

The logarithm of the product of two or more factors is equal to the sum of the logarithms of those factors.

Hence we see that if it is required to multiply two or more numbers by each other, we have only to add their logarithms; the sum will be the logarithm of their product. We then look in the Table for the number answering to that logarithm, in order to obtain the required product.

EXAMPLES.

Ex. 1. Find the product of 8 and 9 by means of logarithms.

In Art. 334, the logarithm of 8 is given	0.903090
" 9 "	0.954243
The sum of these two logarithms is	1.857333

which, according to the same Table, is seen to be the logarithm of 72.

Ex. 2. Find the continued product of 2, 5, and 14 by means of logarithms.

Ex. 3. Find the continued product of 2, 3, 4, and 5 by means of logarithms.

If, instead of multiplying, we divide equation (1) by equation (2), we shall obtain

$$\frac{N}{N'} = \frac{a^x}{a^{x^1}} = a^{x-x^1}.$$

Therefore, according to the definition, $x - x^1$ is the ogarithm of $\frac{N}{N'}$, since $x - x^1$ is the index of that power of the base a which is equal to $\frac{N}{N'}$; that is to say,

The logarithm of a fraction, or of the quotient of one number divided by another, is equal to the logarithm of the numerator, minus the logarithm of the denominator.

Hence we see that if we wish to divide one number by another, we have only to subtract the logarithm of the divisor from that of the dividend; the difference will be the logarithm of their quotient.

EXAMPLES.

Ex. 1. It is required to divide 108 by 12 by means of logarithms.

The logarithm of 108 is	2.033424
" 12	1.079181
The difference is	0.954243,

which is the logarithm corresponding to the number 9.

Ex. 2. Divide 133 by 7 by means of logarithms.

Ex. 3. Divide 136 by 17 by means of logarithms.

Ex. 4. Divide 135 by 15 by means of logarithms.

The preceding examples are designed to illustrate the properties of logarithms. In order to exhibit fully their *utility* in computation, it would be necessary to employ larger numbers; but that would require a more extensive Table than the one given in Art. 334.

(324.) Logarithms are attended with still greater advantages in the involution of powers, and in the extraction of roots. For if we raise both members of equation (1) to the nth power, we obtain

$$N^n = a^{nx}.$$

Therefore, according to the definition, nx is the logarithm of N^n, since nx is the index of that power of the base which is equal to N^n; that is to say,

The logarithm of any power of a number, is equal to the logarithm of that number multiplied by the exponent of the power.

EXAMPLES.

Ex. 1. Find the third power of 4 by means of logarithms.

The logarithm of 4 is	0.602060
Multiply by	3
The product is	1.806180,

which is the logarithm of 64.

Ex. 2. Find the fourth power of 3 by means of logarithms.

Ex. 3. Find the seventh power of 2 by means of logarithms.

Ex. 4. Find the third power of 5 by means of logarithms.

Again, if we extract the nth root of both members of equation (1), we shall obtain

$$N^{\frac{1}{n}} = a^{\frac{x}{n}}$$

therefore, according to the definition, $\frac{x}{n}$ is the logarithm of $N^{\frac{1}{n}}$; that is to say,

The logarithm of any root of a number, is equal to the logarithm of that number divided by the index of the root.

EXAMPLES.

Ex. 1. Find the square root of 81 by means of logarithms.

The logarithm of 81 is	1.908485
Divided by 2	
The quotient is	.954243,

which is the logarithm of 9.

Ex. 2. Find the square root of 121 by means of logarithms.

Ex. 3. Find the sixth root of 64 by means of logarithms.

Ex. 4. Find the third root of 125 by means of logarithms.

The preceding examples will suffice to show, that if we had tables which gave the logarithms of all numbers, they would prove highly useful when we have occasion to perform frequent multiplications, divisions, involutions, and extraction of roots.

(325.) The following examples will show the application of some of the preceding principles.

Ex. 1. $\log. (abcd) = \log. a + \log. b + \log. c + \log. d.$

Ex. 2. $\log. \left(\frac{abc}{de}\right) = \log. a + \log. b + \log. c - \log. d - \log. e.$

Ex. 3. $\log. (a^m b^n c^p) = m \log. a + n \log. b + p \log. c.$

Ex. 4. $\log. \left(\frac{a^m b^n}{c^p}\right) = m \log. a + n \log. b - p \log. c.$

Ex. 5. $\log. (a^2 - x^2) = \log. [(a + x)(a - x)] = \log. (a + x) +$ $\log. (a - x).$

Ex. 6. $\log. \sqrt{a^2 - x^2} = \frac{1}{2} \log. (a + x) + \frac{1}{2} \log. (a - x)$

Ex. 7. $\log. a^3 \sqrt[4]{a^3} = \log. \left(a^{\frac{15}{4}}\right) = \frac{15}{4} \log. a.$

(326.) We shall presently explain a method by which logarithms may be computed. We may observe, however, that it is not necessary to compute the logarithms of all numbers independently. From the logarithms of a few numbers we may readily derive the logarithms of a great many other numbers.

We have seen, in Art. 323, that the logarithm of a product is found by adding together the logarithms of the factors. Let us represent the logarithm of 2 by x; then, since the logarithm of 10 is 1, we shall have

$$\begin{array}{llll} \log. 20 & = x + 1, & \log. 20000 & = \\ \log. 200 & = x + 2, & \log. 200000 & = \\ \log. 2000 & = x + 3, & \log. 2000000 & =, \text{ \&c.} \end{array}$$

We have seen, in Art. 324, that the logarithm of any power of a number is equal to the logarithm of that number multiplied by the exponent of the power.

Hence,

$$\begin{array}{llll} \log. 4 & = 2x, & \log. 32 & = \\ \log. 8 & = 3x, & \log. 64 & = \\ \log. 16 & = 4x, & \log. 128 & =, \text{ \&c.} \end{array}$$

Hence we find, also, that

$$\begin{array}{llll}
\text{log. } 40 & = 2x + 1, & \text{log. } 4000 & = \\
\text{log. } 400 & = 2x + 2, & \text{log. } 40000 & =, \text{ \&c.} \\
\text{log. } 80 & = 3x + 1, & \text{log. } 8000 & = \\
\text{log. } 800 & = 3x + 2, & \text{log. } 80000 & =, \text{ \&c.} \\
\text{log. } 160 & = 4x + 1, & \text{log. } 16000 & = \\
\text{log. } 1600 & = 4x + 2, & \text{log. } 160000 & =, \text{ \&c.}
\end{array}$$

We have seen, in Art. 323, that the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator. Hence, log. $(\frac{10}{2})$ = log. $5 = 1 - x$.

$$\begin{array}{lllll}
\text{Hence,} & \text{log. } 50 & = 2 - x, & \text{log. } 5000 & = \\
 & \text{log. } 500 & = 3 - x, & \text{log. } 50000 & =, \text{ \&c.} \\
 & \text{log. } 25 & = 2 - 2x, & \text{log. } 625 & = \\
 & \text{log. } 125 & = 3 - 3x, & \text{log. } 3125 & =, \text{ \&c.} \\
 & \text{log. } 250 & = 3 - 2x, & \text{log. } 25000 & = \\
 & \text{log. } 2500 & = 4 - 2x, & \text{log. } 250000 & =, \text{ \&c.} \\
 & \text{log. } 1250 & = 4 - 3x, & \text{log. } 125000 & = \\
 & \text{log. } 12500 & = 5 - 3x, & \text{log. } 1250000 & =, \text{ \&c.} \\
 & \text{log. } 6250 & = 5 - 4x, & \text{log. } 625000 & = \\
 & \text{og. } 62500 & = 6 - 4x, & \text{log. } 6250000 & =, \text{ \&c.}
\end{array}$$

(327.) So, also, from the logarithm of 3 we might easily derive a great number of other logarithms. From the Table on page 340, we find the logarithm of 3 to be .477121; it is required to derive from this the logarithm of 30.

Required the logarithm of 3000.
Required the logarithm of 9.
Required the logarithm of 27.
Required the logarithm of 81.
Required the logarithm of 90.
Required the logarithm of 270.
Required the logarithm of 900.

From the same Table, we find the logarithm of 2 to be .301030. It is required, by the aid of the logarithms of 3 and 2, to obtain the logarithm of 6.

Required the logarithm of 12.

Required the logarithm of 15.

Required the logarithm of 18.

From the same Table, we find the logarithm of 5 to be .698970. It is required from this to deduce the logarithm of 50.

Required the logarithm of 500.

Required the logarithm of 5000.

From the same Table we find the logarithm of 95 to be 1.977724. The logarithm of 9.5, or $\frac{95}{10}$, is equal to the logarithm of 95 minus the logarithm of 10.

Hence, the logarithm of 9.5 is 0.977724.

Also, the logarithm of 950 is 2.977724.

Hence the decimal part of the logarithm of any number is the same as that of the number multiplied or divided by 10, 100, 1000, &c.

Prime numbers are such as cannot be decomposed into factors; as, 2, 3, 5, 7, 11, 13, 17, &c. All other numbers arise from the multiplication of prime numbers. If, therefore, we knew the logarithms of all the prime numbers, we could find the logarithms of all other number by simple addition.

(328.) We will now explain a method by which the logarithm of any number may be computed.

If a series of numbers be taken in *Geometrical progression*, their logarithms will form a series in *Arithmetical progression.* Thus, take the geometrical series

1, 10, 100, 1000, 10000, 100000,

their logarithms are

0, 1, 2, 3, 4, 5,

forming an arithmetical series.

If, now, we interpolate a geometrical mean between any two numbers in the first series, its logarithm will be the arithmetical mean between the two corresponding numbers in the lower series.

Find, for example, a geometrical mean between 1 and 10. It will be the square root of 10, or 3.162277. The arithmetical mean between 0 and 1 is 0.5.

Therefore, the logarithm of 3.162277 is 0.5.

Find, again, a geometrical mean between 3.162277 and 10, which is 5.623413. Find, also, the arithmetical mean between 0.5 and 1, which is 0.75.

Therefore, the logarithm of 5.623413 is 0.75.

Find now a geometrical mean between 3.162277 and 5.623413, which is 4.216964. Its logarithm will be the arithmetical mean between 0.5 and 0.75, which is 0.625.

Therefore, the logarithm of 4.216964 is 0.625.

Find, again, a geometrical mean between 4.216964 and 5.623413, which is 4.869674. Its logarithm will be the arithmetical mean between 0.625 and 0.75, which is 0.6875.

Thus we have found the logarithms of four new numbers, and in this manner we might proceed to construct a table of logarithms. It will be observed, however, that these numbers are all *fractional*, whereas it is most convenient to have the logarithms of *integers*. By pursuing this method, however, we might eventually find the logarithm of a whole number; as, for example, 5. For we have already found the logarithm of

5.623413 to be 0.75,
and the logarithm of 4.869674 " 0.6875.

One of these numbers is *greater* than 5, and the other *less*. A geometrical mean between them is 5.232991, which is too great; but the mean between this result and the last of the two preceding is 5.048065, which is already a close approximation. By pursuing the same method, we may come nearer and nearer to the number 5, until at last, after finding twenty-two geometrical means, the difference is inappreciable in the sixth decimal place, and we obtain

the logarithm of 5 equal to 0.69897;

and by a like process, the logarithm of any other number may be found.

(329.) Hence, to compute the logarithm of any number, we have the following

RULE.

Take the geometrical series 1, 10, 100, 1000, 10000, &c., *and apply to it the arithmetical series* 0, 1, 2, 3, 4, &c., *as logarithms.*

Find a geometrical mean between 1 *and* 10, 10 *and* 100, *or any other two terms of the first series between which the proposed number lies.*

Between the mean thus found and the nearest extreme, find another geometrical mean in the same manner, and so on, till you arrive within the proposed limit of the number whose logarithm is sought.

Find, also, as many arithmetical means between the corresponding terms 0, 1, 2, 3, 4, &c., *of the other series, in the same order as you found the geometrical ones ; the last of these will be the logarithm answering to the number required.*

In this manner were the logarithms of all the prime numbers at first computed ; but much more expeditious methods have since been devised.

Having obtained the logarithm of 5, it is easy to find the logarithm of 2. For the logarithm of $2 = \log. (\frac{10}{5}) = \log. 10 - \log. 5 = 1 - 0.69897 = 0.30103$.

(330.) We have seen, in Art. 322, that

the logarithm of		0.1	is -1,
"	"	0.01	" -2,
"	"	0.001	" -3,
"	"	0.0001	" -4,
"	"	0.00001	" -5, &c.

That is, the smaller the fraction, the greater its logarithm. Hence, if the fraction be infinitely small, its logarithm will be infinitely great ; that is, in Briggs' system, *the logarithm of* 0 *is infinite and negative.*

LOGARITHMIC SERIES.

(331.) We have already explained a method of computing logarithms ; but this method is very laborious in practice. It is found much more convenient to express the logarithm of a number in the form of a *series.*

Let y be a number whose logarithm is required to be developed in a series, and let us employ the method of Unknown Coefficients. Put $y = 1 + x$, and assume

$$\log. (1 + x) = Ax + Bx^2 + Cx^3 + Dx^4 +, \&c., \quad (1).$$

Assume, also,

$$\log.(1+z) = Az + Bz^2 + Cz^3 + Dz^4 +, \&c. \quad (2).$$

Subtracting equation (2) from (1), we obtain

$$\log.(1+x) - \log.(1+z) = A(x-z) + B(x^2-z^2) + C(x^3-z^3) +, \&c. \quad (3).$$

The second member of this equation is divisible by $x-z$; we will reduce the first member to a form in which it shall also be divisible.

We have $\log.(1+x) - \log.(1+z) = \log.\left(\frac{1+x}{1+z}\right) = \log.\left(1+\frac{x-z}{1+z}\right)$.

But, since $\frac{x-z}{1+z}$ may be regarded as a single quantity, v, we may develop $\log.(1+v)$ in the same manner as $\log.(1+x)$, which gives

$$\log.\left(1+\frac{x-z}{1+z}\right) = A.\frac{x-z}{1+z} + B\left(\frac{x-z}{1+z}\right)^2 + C\left(\frac{x-z}{1+z}\right)^3 +, \&c.$$

This last series must be identical with the one which we have already obtained for $\log.\left(1+\frac{x-z}{1+z}\right)$, or its equal, $\log.(1+x) - \log.(1+z)$, in equation (3); and since the terms of both are divisible by $x-z$, by taking out this common factor, we obtain

$$A.\frac{1}{1+z} + B\frac{x-z}{(1+z)^2} + C\frac{(x-z)^2}{(1+z)^3} +, \&c., = A + B(x+z) + C(x^2 + xz + z^2) +, \&c.$$

Since this equation, like the preceding, must be verified for all values of x and z, the equality must subsist when $x = z$. But on this hypothesis, all the terms of the first series vanish except one, and we have simply

$$\frac{A}{1+x} = A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4 +, \&c.;$$

or, performing the division indicated in the first member,

$$A(1 - x + x^2 - x^3 + x^4 - \ldots) = A + 2Bx + 3Cx^2 + 4Dx^3 + \ldots$$

Therefore, according to the principle of Art. 287, we have the equations

$$A = A,$$

$$-A = 2B; \text{ whence } B = -\frac{A}{2}.$$

$$A = 3C; \quad \text{"} \quad C = +\frac{A}{3}.$$

$$-A = 4D \quad \text{"} \quad D = -\frac{A}{4}.$$

$$A = 5E; \quad \text{"} \quad E = +\frac{A}{5}$$

The law of the series is obvious; and hence, substituting the values of B, C, D, &c., in equation (1), we obtain, for the development of log. $(1+x)$,

$$\log.\,(1+x) = \frac{A}{1}\cdot x - \frac{A}{2}x^2 + \frac{A}{3}x^3 - \frac{A}{4}x^4 + \dots$$

$$= A\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots\right).$$

The number A is called the *modulus* of the system of logarithms employed. Lord Napier, the illustrious inventor of logarithms, assumed the modulus equal to unity. If, then, we designate Naperian logarithms by log.′, we shall have

$$\log.'\,(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} +, \text{\&c.} \quad (4).$$

By giving to x in succession all possible values, we may obtain from this equation the logarithms of all numbers.

If we make $x = 0$, we shall have log.′ $1 = 0$.

Make $x = 1$, and we obtain

$$\log.'\,2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} -, \text{\&c.},$$

a series which converges so slowly that it would be necessary to employ a very large number of terms to obtain the accuracy desirable. The series may be rendered more converging in the following manner:

In equation (4), substitute $-x$ for x, and it becomes

$$\log.'\,(1-x) = -\frac{x}{1} - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} -, \text{\&c.} \quad (5).$$

Subtracting equation (5) from equation (4), and observing that $\log.' (1+x) - \log.' (1-x) = \log.' \frac{1+x}{1-x}$, we obtain

$$\log.' \left(\frac{1+x}{1-x}\right) = 2\left(\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \ldots\right).$$

Put $\frac{1+x}{1-x} = 1 + \frac{1}{z}$ (z being at least equal to unity), and we have $(1+x)z = (1-x)(z+1)$; whence, by reduction, $x = \frac{1}{2z+1}$.

Hence the preceding series becomes

$$\log.' \left(1+\frac{1}{z}\right) = \log.' (z+1) - \log.' z = 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \ldots\right).$$

(332.) The last series may be employed for computing the logarithm of any number, when the logarithm of the preceding number is known. Making successively $z = 1, 2, 4, 6$, &c., we find the following

NAPERIAN, OR HYPERBOLIC LOGARITHMS.

$\log.'\ 2 = 2(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \ldots) = 0.693147$

$\log.'\ 3 = \log.' 2 + 2(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \frac{1}{7 \cdot 5^7} + \ldots) = 1.098612$

$\log.'\ 4 = 2 \log.' 2 = 1.386294$

$\log.'\ 5 = \log.' 4 + 2(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \frac{1}{7 \cdot 9^7} + \ldots) = 1.609438$

$\log.'\ 6 = \log.' 3 + \log.' 2 = 1.791759$

$\log.'\ 7 = \log.' 6 + 2(\frac{1}{13} + \frac{1}{3 \cdot 13^3} + \frac{1}{5 \cdot 13^5} + \frac{1}{7 \cdot 13^7} + \ldots) = 1.945910$

$\log.'\ 8 = \log.' 4 + \log.' 2 = 2.079442$

$\log.'\ 9 = 2 \log.' 3 = 2.197225$

$\log.'\ 10 = \log.' 5 + \log.' 2 = 2.302585$

&c., &c., &c.

(333.) The Naperian logarithms being computed, it is easy to form any other system. We have found

$$\log. (1+x) = A\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots\right).$$

Distinguishing the Naperian logarithms by an accent, we have

$$\log.'\,(1+x) = \mathrm{A}'\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} \cdots\right).$$

Hence

$$\log.\,(1+x) : \log.'\,(1+x) : : \mathrm{A} : \mathrm{A}'.$$

Therefore, *the logarithms of the same number in different systems are to each other as the moduli.*

In Napier's system, the modulus $= 1$. Hence

$$\log.\,(1+x) = \mathrm{A} \,.\, \log.'\,(1+x).$$

That is, *the common logarithm of a number is equal to its Naperian logarithm multiplied by the modulus of the common system.*

If, then, we knew the modulus of the common system, we could easily convert the preceding Naperian logarithms into common logarithms. Now, from the equation

$$\log.\,(1+x) = \mathrm{A} \,.\, \log.'\,(1+x), \text{ we obtain}$$

$$\mathrm{A} = \frac{\log.\ (1+x)}{\log.'\,(1+x)}.$$

Suppose $x = 9$, then $\mathrm{A} = \dfrac{\log.\ 10}{\log.'\ 10}$.

But $\log.\ 10 = 1$. Hence

$$\mathrm{A} = \frac{1}{\log.'\ 10} = \frac{1}{2.302585} = 0.434294,$$

which is the modulus of the common system.

(334.) We can now compute the

COMMON, OR BRIGGS' LOGARITHMS.

$\log.\ 2 = 0.434294 \times 0.693147 = 0.301030$

$\log.\ 3 = 0.434294 \times 1.098612 = 0.477121$

$\log.\ 4 = 2 \log.\ 2 = 0.602060$

$\log.\ 5 = \log.\ 10 - \log.\ 2 = 1 - \log.\ 2 = 0.698970$

$\log.\ 6 = \log.\ 3 + \log.\ 2 = 0.778151$

$\log.\ 7 = 0.434294 \times 1.945910 = 0.845098$

$\log.\ 8 = 3 \log.\ 2 = 0.903090$

$\log.\ 9 = 2 \log.\ 3 = 0.954243$

$\log.\ 10 = \quad = 1.000000$

&c., &c., &c.

We thus obtain the following Table of Common Logarithms:

No.	Logarithm.	No.	Logarithm.	No.	Logarithm.	No.	Logarithm.
1	0.000000	36	1.556303	71	1.851258	106	2.025306
2	0.301030	37	1.568202	72	1.857333	107	2.029384
3	0.477121	38	1.579784	73	1.863323	108	2.033424
4	0.602060	39	1.591065	74	1.869232	109	2.037426
5	0.698970	40	1.602060	75	1.875061	110	2.041393
6	0.778151	41	1.612784	76	1.880814	111	2.045323
7	0.845098	42	1.623249	77	1.886491	112	2.049218
8	0.903090	43	1.633468	78	1.892095	113	2.053078
9	0.954243	44	1.643453	79	1.897627	114	2.056905
10	1.000000	45	1.653213	80	1.903090	115	2.060698
11	1.041393	46	1.662758	81	1.908485	116	2.064458
12	1.079181	47	1.672098	82	1.913814	117	2.068186
13	1.113943	48	1.681241	83	1.919078	118	2.071882
14	1.146128	49	1.690196	84	1.924279	119	2.075547
15	1.176091	50	1.698970	85	1.929419	120	2.079181
16	1.204120	51	1.707570	86	1.934498	121	2.082785
17	1.230449	52	1.716003	87	1.939519	122	2.086360
18	1.255273	53	1.724276	88	1.944483	123	2.089905
19	1.278754	54	1.732394	89	1.949390	124	2.093422
20	1.301030	55	1.740363	90	1.954243	125	2.096910
21	1.322219	56	1.748188	91	1.959041	126	2.100371
22	1.342423	57	1.755875	92	1.963788	127	2.103804
23	1.361728	58	1.763428	93	1.968483	128	2.107210
24	1.380211	59	1.770852	94	1.973128	129	2.110590
25	1.397940	60	1.778151	95	1.977724	130	2.113943
26	1.414973	61	1.785330	96	1.982271	131	2.117271
27	1.431364	62	1.792392	97	1.986772	132	2.120574
28	1.447158	63	1.799341	98	1.991226	133	2.123852
29	1.462398	64	1.806180	99	1.995635	134	2.127105
30	1.477121	65	1.812913	100	2.000000	135	2.130334
31	1.491362	66	1.819544	101	2.004321	136	2.133539
32	1.505150	67	1.826075	102	2.008600	137	2.136721
33	1.518514	68	1.832509	103	2.012837	138	2.139879
34	1.531479	69	1.838849	104	2.017033	139	2.143015
35	1.544068	70	1.845098	105	2.021189	140	2.146128

(335.) Let us now determine the *base* of Napier's system. Designating it by a, we shall have, Art. 333,

$$\log.' a : \log. a :: 1 : 0.434294.$$

But $\log.' a = 1$. Hence

$$\log. a = 0.434294.$$

That is, *the modulus of the common system is equal to the common logarithm of Napier's base.*

We wish, then, to find the number corresponding to the common logarithm 0.434294. By inspecting the preceding table, we see that this number must be a little less than 3. More accurately, it is

$$2.718282,$$

which is the base of Napier's system.

Any number, except unity, may be taken as the base of a system of logarithms, and hence there may be an infinite number of systems. Only two systems, however, are much used; those of Briggs and Napier.

The base of	Briggs'	system is	10.
"	Napier's	"	2.718282.
The modulus of	Briggs'	"	0.434294.
"	Napier's	"	1.

Hence, in Briggs' system, all numbers are to be regarded as powers of 10

Thus,
$$10^{0.301} = 2,$$
$$10^{0.477} = 3,$$
$$10^{0.602} = 4,$$
$$10^{0.698} = 5.$$
&c., &c.

In Napier's system, all numbers are to be regarded as powers of 2.718282.

Thus,
$$2.718^{0.693} = 2,$$
$$2.718^{1.098} = 3,$$
$$2.718^{1.386} = 4,$$
$$2.718^{1.609} = 5.$$
&c., &c.

Briggs' logarithms are employed in all the common operations of multiplication and division, and hence they are known by the name of *common* logarithms. Napier's logarithms are of great use in the application of the calculus to many analytical and physical problems. They are also called *hyperbolic* logarithms, having been originally derived from the hyperbola.

EXPONENTIAL EQUATIONS.

(336.) An *exponential quantity* is one which is raised to some unknown power, or which has an unknown quantity for an exponent; as,

$$a^x,\ a^{\frac{1}{x}},\ x^x, \text{ or } x^{\frac{1}{x}}, \text{ \&c.}$$

An *exponential equation* is one which contains an exponential quantity; as,

$$a^x = b,\ x^x = a, \text{ \&c.}$$

Such equations are most easily solved by means of logarithms. Thus, consider the equation

$$a^x = b.$$

Taking the logarithm of each member of the equation, we have

$$x \log.\, a = \log.\, b,$$

$$\text{or } x = \frac{\log.\, b}{\log.\, a}.$$

Ex. 1. What is the value of x in the equation $3^x = 81$?

By the preceding formula, $x = \dfrac{\log.\, 81}{\log.\, 3}$.

Looking out the logarithms of 81 and 3 from the Table in Art. 334, we have

$$x = \frac{1.908485}{.477121} = 4.$$

Therefore, $3^4 = 81.$

Ex. 2. What is the value of x in the equation $3^x = 20$?

$$x = \frac{\log.\, 20}{\log.\, 3} = \frac{1.301030}{.477121} = 2.727 \text{ nearly.}$$

Therefore, $3^{2.727} = 20$ nearly.

(337.) The other equation, $x^x = a$ may be solved by *trial*, as in Art. 319. Thus taking the logarithm of each member, we have

$$x \log. x = \log. a.$$

Find now, by trial, two numbers nearly equal to the value of x, and substitute them in the given equation instead of the unknown quantity. Then say,

As the difference of these results,
Is to the difference of the two assumed numbers,
So is the error of either result,
To the correction required in the corresponding assumed number.

Ex. 1. Given $x^x = 100$ to find the value of x.

Here we have $\quad x \log. x = \log. 100 = 2.$

Suppose $\quad x = 3,$

Then $\quad 0.477121 \times 3 = 1.431363$, which is *too small.*

Suppose $\quad x = 4,$

Then $\quad 0.602060 \times 4 = 2.408240$, which is *too great.*

Hence the value of x is between 3 and 4, but nearer to 4. Assume, then, 3.5 and 3.6 for the two numbers.

By the first supposition,		*By the second supposition,*	
$x = 3.5$; $\log. x =$	.544068	$x = 3.6$; $\log. x =$	.556303
Multiplied by	3.5	Multiplied by	3.6
$x . \log. x =$	1.904238	$x . \log. x =$	2.002689

Diff. of results :	*Diff. assumed numbers* : :	*Error of 2d result* :	*Its correction.*
.098451 :	0.1 : :	.002689 :	.00273

Hence $x \quad = 3.6 - .00273 = 3.59727$ nearly.

Therefore, $\quad 3.59727^{3.59727} = 100$ nearly.

If we wish a more accurate result, the operation must be repeated with two new numbers; as, for example, 3.59727 and 3.59728.

Ex. 2. Given $x^x = 20x$, to find an approximate value of x.

COMPOUND INTEREST.

(338.) In calculating compound interest, the first subject of inquiry is, to what sum does a given principal amount, after a certain number of years, the interest being annually added to the principal? It is evident that $1.00, placed out at 5 per cent., becomes at the end of a year, a principal of $1.05. But the amount at the end of each year must be proportioned to the principal at the beginning of the year. In order, then, to find the amount at the end of two years, we say

$$1.00 : 1.05 :: 1.05 : (1.05)^2.$$

To find the amount at the end of three years, we say

$$1.00 : 1.05 :: (1.05)^2 : (1.05)^3.$$

And in the same manner we find that the amount of $1.00 for n years at compound interest is $(1.05)^n$.

If the rate of interest were six per cent., we should find the amount for n years to be $(1.06)^n$.

The amount of two dollars for a given time must obviously be double the amount of one dollar, and the amount of $1000 must be a thousand times the amount of one dollar.

Hence, if we put P to represent the principal,
r the rate per cent. considered as a decimal,
n the number of years,
A the amount of the given principal for n years,

we shall have

$$A = P.(1 + r)^n.$$

This equation contains four quantities, A, P, n, r; any three of which being given, the other may be found. The computations are most readily performed by means of logarithms. Taking the logarithms of both members of the preceding equation and reducing, we find

1. $\text{Log. A} = n \times \log.(1 + r) + \log.\text{P}.$
2. $\text{Log. P} = \log.\text{A} - n \times \log.(1 + r).$
3. $\text{Log.}(1 + r) = \dfrac{\log.\text{A} - \log.\text{P}}{n}.$
4. $n = \dfrac{\log.\text{A} - \log.\text{P}}{\log.(1 + r)}.$

EXAMPLES.

Ex. 1. What is the amount of twenty dollars, at 6 per cent compound interest, for 11 years?

Here we employ formula (1).

Amount of $1.00 for 1 year $1.06,	log. = 0.025306
Multiplying by 11,	11
	0.278366
Given principal $20.	log. = 1.301030
Amount $38 nearly,	1.579396

This result is derived from the Table on page 340. By consulting a larger Table, we should find the amount $37.966.

Ex. 2. What principal at 5 per cent. interest will amount to $50 in 13 years?

Here we employ formula (2).

$1+r=1.05$,	log. = 0.021189
Multiplying by n,	13
Subtract	0.275457
From log. A	= 1.698970
P = $26.5 nearly,	1.423513

More accurately, P = $26.516.

Ex. 3. At what rate per cent. must $40 be put out at compound interest, that it may amount to $57 in 9 years?

Here we employ formula (3).

A = 57,	log. = 1.755875
P = 40,	log. = 1.602060
Dividing by n,	9)0.153815
$1+r=1.04$	= 0.017091

Consequently, $r=.04$, or four per cent.

Ex. 4. In what time will $50 amount to $90 at 5 per cent.

Here we employ formula (4).

A = 90,	log. = 1.954243
P = 50,	log. = 1.698970
$1+r=1.05$, whose logarithm is	0.021189)0.255273

Dividing one logarithm by the other, we obtain 12, Ans.

Ex. 5. What is the amount of $52 at 3 per cent. compound interest for 15 years?

Ans. $81.

Ex. 6. What principal at 6 per cent. compound interest will amount to $101 in 4 years?

Ans. $80.

Ex. 7. At what rate will $10 amount to $16 in 16 years?

Ans. Three per cent.

Ex. 8. What will $300 amount to in 10 years at compound interest semi-annually, the yearly rate being 6 per cent.?

Ex. 9. In what time will a sum of money double at 6 per cent. compound interest?

Ans. 11.89 years.

Ex. 10. In what time will a sum of money triple itself at 4 per cent. compound interest?

Ans. 28.01 years.

(339.) The natural increase of population in a country may be computed in the same way as compound interest. Knowing the population at two different dates, we compute the rate of increase by formula (3), and from this we may compute the population at any future time on the supposition of a uniform rate of increase.

EXAMPLES.

Ex. 1. The number of the inhabitants of the United States in 1790 was 3,900,000; and in 1840, 17,000,000. What was the average increase for every ten years?

Ans. 34 per cent.

Ex. 2. Suppose the rate of increase to remain the same for the next ten years, what would be the number of inhabitants in 1850?

Ans. 22,800,000.

Ex. 3. At the same rate, in what time would the number in 1840 be doubled?

Ans. 23.54 years.

Ex. 4. At the same rate, what was the population in 1780?

Ans. 2,900,000.

Ex. 5. At the same rate, in what time would the number in 1840 be tripled?

Ans. 37.31 years.

THE END.

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